

Multiuser Broadcast Erasure Channel with Feedback — Capacity and Algorithms

Marios Gatzianas, *Member, IEEE*, Leonidas Georgiadis, *Senior Member, IEEE*, and
Leandros Tassioulas, *Fellow, IEEE*

Abstract

We consider the N -user broadcast erasure channel with N unicast sessions (one for each user) where receiver feedback is regularly sent to the transmitter in the form of ACK/NACK messages. We first provide a generic outer bound to the capacity of this system; we then propose a virtual-queue-based inter-session mixing coding algorithm, determine its rate region and show that it achieves capacity under certain conditions on channel statistics, assuming that instantaneous feedback is known to all users. Removing this assumption results in a rate region that asymptotically differs from the outer bound by 1 bit as $L \rightarrow \infty$, where L is the number of bits per packet (packet length). For the case of arbitrary channel statistics, we present a modification of the previous algorithm whose rate region is identical to the outer bound for $N = 3$, when instant feedback is known to all users, and differs from the bound by 1 bit as $L \rightarrow \infty$, when the 3 users know only their own ACK. The proposed algorithms do not require any prior knowledge of channel statistics.

Index Terms

Broadcast erasure channels, unicast traffic, feedback-based coding, capacity achieving algorithms.

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M. Gatzianas is with the Center for Research and Technology Hellas, Informatics & Telematics Institute (CERTH-ITI), 6th km Charilaou-Thermi road, Thessaloniki, 57 001, Greece (e-mail: mgatzia@ee.auth.gr).

L. Georgiadis is with the Department of Electrical and Computer Engineering, Division of Telecommunications, Aristotle University of Thessaloniki, Thessaloniki, 54 124, Greece and with CERTH-ITI (e-mail: leonid@auth.gr).

L. Tassioulas is with the Computer Engineering and Telecommunications Department, University of Thessaly, Volos, 38 221, Greece and with CERTH-ITI (e-mail: leandros@uth.gr).

I. INTRODUCTION

Broadcast channels have been extensively studied by the information theory community since their introduction in [1]. Although their capacity remains unknown in the general case, special cases have been solved, including the important category of “degraded” channels [2]. Another class of channels that has received significant attention is erasure channels, where either the receiver receives the input symbol unaltered or the input symbol is erased (i.e. not received at all) at the receiver. The class of erasure channels is usually employed as a model for lossy packet networks.

Combining the above classes, a broadcast packet erasure channel (BPEC) is a suitable abstraction for wireless communications modeling since it captures the essentially broadcast nature of the medium as well as the potential for packet loss (due to fading, packet collision etc). Since this channel is not necessarily degraded, the computation of its feedback capacity region is an open problem. Numerous variations of this channel, under different assumptions, have been studied, a brief summary of which follows.

For multicast traffic, an outer bound to the capacity region of erasure channels is derived in [3], in the form of a suitably defined minimum cut, and it is proved that the bound can be achieved by linear coding at intermediate nodes. The broadcast nature is captured by requiring each node to transmit the same signal on all its outgoing links, while it is assumed that the destinations have complete knowledge of any erasures that occurred on all source-destination paths. In a sense, [3] is the “wireless” counterpart to the classical network coding paradigm of [4], since it carries all results of [4] (which were based on the assumption of error-free channels) into the wireless regime.

The concept of combining packets for efficient transmission based on receiver feedback is also used in [5], where broadcast traffic is assumed and a rate-optimal, zero-delay, offline algorithm is presented for 3 users. Online heuristics that attempt to minimize the decoding delay are also presented. Reference [6] expands on this work by presenting an online algorithm that solves at each slot a (NP-hard) set packing problem in order to decide which packets to combine. This algorithm also aims in minimizing delay.

Multiple unicast flows, which are traditionally difficult to handle within the network coding paradigm, are studied in [7] for a network where each source is connected to a relay as well as to all destinations, other than its own, and all connections are modeled as BPECs. A capacity outer bound is presented for an arbitrary number of users N and is shown to be achievable for $N = 3$ and almost achievable for $N = 4, 5$. The capacity-achieving algorithm operates in two stages with the relay having knowledge of the destination message side information at the end of the first stage but not afterward (i.e. once the

second stage starts, the relay does not receive feedback from the destinations).

A similar setting is studied in [8], where ACK-based packet combining is proposed and emphasis is placed on the overhead and complexity requirements of the proposed scheme. An actual implementation of packet XORing in an intermediate layer between the IP and 802.11 MAC layers is presented and evaluated in [9], while [10] proposes a replacement for the 802.11 retransmission scheme based on exploiting knowledge of previously received packets.

This paper expands upon earlier work in [11], [12], (which studied the case $N = 2$) and differs from the aforementioned works in that, although it also uses the idea of packet mixing (in the network coding sense), it introduces additional concepts and tools that generalize the results concerning achievable rates to more than 2 users and provides explicit performance guarantees. Specifically, an outer bound to the feedback capacity region for multiple unicast flows (one for each user) is computed and, assuming public feedback is available, two online algorithms (named CODE1_{pub} and CODE2_{pub}) are presented that achieve this bound under certain conditions on rates and channel statistics. If public feedback is not available, we propose modifications to these algorithms that achieve rates within 1 bit/transmission of the outer bound asymptotically in the size of packet length.

The algorithms do not require any knowledge of channel parameters (such as erasure probabilities) or future events so that they can be applied to any BPEC. They use receiver feedback to combine packets intended for different users into a single packet which is then transmitted. The combining scheme (i.e. choosing which packets to combine and how) relies on a group of virtual queues, maintained in the transmitter, which are updated based on per-slot available receiver ACK/NACKs. This queue-based coding concept has also been used in [13], albeit for broadcast traffic with stochastic arrivals where the stability region of the proposed algorithm becomes asymptotically optimal as the erasure probability goes to 0, whereas we consider systems with an arbitrarily fixed number of packets per unicast session where the capacity is achieved for arbitrary values of erasure probability.

During the preparation of this paper, we were informed that C. Wang has independently studied in [14] the same problem as appears here and proposed coding algorithms that achieve capacity under the same conditions as ours. Although the two works share common ideas (namely, employing degraded channels to derive capacity outer bounds and performing packet coding based on receiver feedback), the proposed algorithms, the procedures for handling overhead, as well as the methodology used for deriving their rate regions, are quite different.

The paper is structured as follows. Section II describes the exact model under investigation and provides the necessary definitions in order to derive the capacity outer bound in Section III. The first coding

algorithm is presented in Section IV, along with a discussion of the intuition behind the algorithm and a detailed example. The main properties of the algorithm are also presented. The algorithm's optimal performance under certain conditions on channel statistics and publicly available feedback is established in Section V. We also present a variant of the algorithm that does not require public feedback due to the incorporation of overhead and determine the corresponding reduction. A modification of the algorithm that achieves capacity for 3 users under arbitrary channel statistics is presented in Section VI, while Section VII concludes the paper. Appendices A–G contain most of the technical proofs.

II. SYSTEM MODEL AND DEFINITIONS

The system model is a direct extension to N users of the corresponding model in [11] but is nonetheless repeated for completeness. We study a time-slotted system where a packet of fixed length L bits is transmitted in each slot. Without loss of generality, we normalize to unity the actual time required to transmit a single bit so that the time interval $[(l-1)L, lL)$, for $l = 1, 2, \dots$, corresponds to slot l . The communication system consists of a single transmitter and a set $\mathcal{N} \triangleq \{1, 2, \dots, N\}$ of receivers/users (we hereafter use these two terms interchangeably), while the channel is modeled as memoryless broadcast erasure (BE), so that each broadcast packet is either received unaltered by a user or is “erased” (i.e. the user does not receive the packet). The latter case is equivalent to considering that the user receives the special symbol E , which is distinct from any other possible transmitted packet and does not actually map to a physical packet (since it models an erasure). We hereafter use the term “packet” to refer to any sequence of L bits and the term “symbol” to refer to a packet or an erasure E (we retain however the standard nomenclature of “input symbol” and “output symbol”, although the former is a true packet while the latter can also be an erasure).

In information-theoretic terms, the broadcast packet erasure channel is described by the tuple $(\mathcal{X}, (\mathcal{Y}_i : i \in \mathcal{N}), p(\mathbf{Y}_l | X_l))$, where \mathcal{X} is the input symbol alphabet (we hereafter assume $\mathcal{X} = \mathbb{F}_q$, with \mathbb{F}_q a suitable field of size q), $\mathcal{Y}_i = \mathcal{Y} = \mathcal{X} \cup \{E\}$ is the output symbol alphabet (where $E \notin \mathcal{X}$) for user i , and $p(\mathbf{Y}_l | X_l)$ is the probability of having, at slot l , output $\mathbf{Y}_l \triangleq (Y_{i,l}, i \in \mathcal{N})$ for a broadcast input symbol X_l . The memoryless property implies that $p(\mathbf{Y}_l | X_l)$ is independent of l , so that it is simply written as $p(\mathbf{Y} | X)$. Since the transmitted symbols are packets of L bits, we identify \mathbb{F}_q with the set of L -bit sequences, so that it holds $q = 2^L$.

Define $Z_{i,l} \triangleq \mathbb{I}[Y_{i,l} = E]$ as the indicator function of an erasure occurring for user i at slot l , and consider the random vector $\mathbf{Z}_l = (Z_{i,l}, i \in \mathcal{N})$. The sequence $\{\mathbf{Z}_l\}_{l=1}^\infty$ is assumed to consist of temporally iid vectors (we denote with \mathbf{Z} the random vector with distribution equal to that of \mathbf{Z}_l), although, for a

fixed slot, arbitrary correlation between erasures for different users is allowed. For any index set $\mathcal{I} \subseteq \mathcal{N}$, we define $E_{\mathcal{I}} \triangleq \{Z_i = 1, \forall i \in \mathcal{I}\} = \cap_{i \in \mathcal{I}} \{Z_i = 1\}$ as the event that an erasure occurs for *all* users in \mathcal{I} . We also use the convention that an intersection over an empty index set yields the entire space to define $E_{\emptyset} \triangleq \Omega$ (the sample space). We denote $\epsilon_{\mathcal{I}} \triangleq \Pr(E_{\mathcal{I}})$ (so that $\epsilon_{\emptyset} = 1$) and, for simplicity, write ϵ_i instead of $\epsilon_{\{i\}}$. In order to avoid trivially degenerate cases, we henceforth assume $\epsilon_i < 1$ for all $i \in \mathcal{N}$.

Using the introduced notation, when the transmitter at the beginning of slot l broadcasts symbol X_l , each user i receives symbol $Y_{i,l} = Z_{i,l}E + (1 - Z_{i,l})X_l$. At the end of each slot l , all users inform the transmitter whether the symbol was received or not, which is equivalent to each user i sending the value of $Z_{i,l}$ (essentially, a simple ACK/NACK) through an error-free zero-delay control channel.

A channel code, denoted as (M_1, \dots, M_N, n) , for the broadcast channel with feedback is now defined as the aggregate of the following components (this is an extension of the standard definition in [15] to N users):

- message sets \mathcal{W}_i of size $|\mathcal{W}_i| = M_i$ for each user $i \in \mathcal{N}$, where $|\cdot|$ denotes set cardinality. Denote the message that needs to be communicated as $\mathbf{W} \triangleq (W_i, i \in \mathcal{N}) \in \mathcal{W}$, where $\mathcal{W} \triangleq \mathcal{W}_1 \times \dots \times \mathcal{W}_N$. It will also be helpful to interpret the message set \mathcal{W}_i as follows: assume that user i needs to decode a given set \mathcal{K}_i of L -bit packets. Then, \mathcal{W}_i is the set of all possible $|\mathcal{K}_i|L$ bit sequences, so that it holds $|\mathcal{W}_i| = M_i = 2^{|\mathcal{K}_i|L}$.
- an encoder that transmits, at slot l , a symbol $X_l = f_l(\mathbf{W}, \mathbf{Y}^{l-1})$ belonging to \mathbb{F}_q , based on the value of \mathbf{W} and all previously gathered feedback $\mathbf{Y}^{l-1} \triangleq (\mathbf{Y}_1, \dots, \mathbf{Y}_{l-1})$. X_1 is a function of \mathbf{W} only. A total of n symbols are transmitted for message \mathbf{W} .
- N decoders, one for each user $i \in \mathcal{N}$, represented by the decoding functions $g_i : \mathcal{Y}^n \rightarrow \mathcal{W}_i$, so that the reconstructed symbol is $\hat{W}_i = g_i(Y_i^n)$, where $Y_i^n \triangleq (Y_{i,1}, \dots, Y_{i,n})$ is the sequence of symbols received by user i (including any erasure symbols E) during the n slots. Thus, the decoding performed by user i depends only on packets received by i , i.e. each user knows only its own feedback.

Hence, a code \mathcal{C} is fully specified by the tuple $(M_1, \dots, M_N, n, (f_l : l = 1, \dots, n), (g_i : i \in \mathcal{N}))$, which contains the message set size along with the encoding/decoding functions; for brevity, we will simply write (M_1, \dots, M_N, n) to denote \mathcal{C} . The probability of erroneous decoding for message \mathbf{W} is $\lambda_n(\mathbf{W}) = \Pr(\cup_{i \in \mathcal{N}} \{g_i(Y_i^n) \neq W_i\} | \mathbf{W})$. The rate \mathbf{R} for this code, measured in information bits per transmitted symbol, is now defined as the vector $\mathbf{R} = (R_i : i \in \mathcal{N})$ with $R_i = (\log_2 M_i)/n$. Hence, it holds $R_i = |\mathcal{K}_i|L/n$.

Let \mathfrak{C} be a class of (M_1, \dots, M_N, n) codes. Then, a vector rate $\mathbf{R} = (R_1, \dots, R_N)$ is achievable *under*

\mathfrak{C} if there exists a sequence of codes $(\lceil 2^{nR_1} \rceil, \dots, \lceil 2^{nR_N} \rceil, n)$ in \mathfrak{C} such that $\frac{1}{|\mathcal{W}|} \sum_{\mathbf{W} \in \mathcal{W}} \lambda_n(\mathbf{W}) \rightarrow 0$ as $n \rightarrow \infty$. Equivalently, we say that \mathfrak{C} achieves rate \mathbf{R} . The closure of the set of rates \mathbf{R} that are achievable under \mathfrak{C} constitutes the rate region of \mathfrak{C} . We further define a rate \mathbf{R} to be achievable if there exists some class \mathfrak{C} of codes that achieves \mathbf{R} . Finally, the capacity region of a channel is defined as the closure of the set of all achievable rates, i.e. the closure of the union of rate regions of all possible classes of codes \mathfrak{C} for this channel.

The following definition, introduced in [2], will be useful in deriving the outer bound for the capacity of the broadcast erasure channel.

Definition 1: A broadcast, not necessarily erasure, channel $(\mathcal{X}, (\mathcal{Y}_i : i \in \mathcal{N}), p(\mathbf{Y}|X))$ with receiver set \mathcal{N} is physically degraded if there exists a permutation $\hat{\pi}$ on \mathcal{N} such that the sequence $X \rightarrow Y_{\hat{\pi}(1)} \rightarrow \dots \rightarrow Y_{\hat{\pi}(N)}$ forms a Markov chain.

A generalization to N users of the 2-user proof in [16] provides the following result.

Lemma 1: Feedback does not increase the capacity region of a physically degraded broadcast channel. We now have all necessary tools to compute a capacity outer bound.

III. CAPACITY OUTER BOUND

Our derivation of the capacity outer bound is based on a method similar to the approaches in [14], [17]–[19]. We initially state a general result on the capacity of broadcast erasure channels *without feedback* [20].

Lemma 2: The capacity region (measured in information bits per transmitted symbol) of a broadcast erasure channel with receiver set \mathcal{N} and no feedback is

$$\mathcal{C}_{noFB} = \left\{ \mathbf{R} \geq \mathbf{0} : \sum_{i \in \mathcal{N}} \frac{R_i}{1 - \epsilon_i} \leq L \right\}, \quad (1)$$

which implies that capacity can be achieved by a simple timesharing scheme.

We denote with C the channel under consideration and, for an arbitrary permutation π on \mathcal{N} , introduce a new, hypothetical, broadcast channel \hat{C}_π with the same input/output alphabets as C and an erasure indicator function of $\hat{Z}_{\pi(i),l} = \prod_{j=1}^i Z_{\pi(j),l}$. In other words, a symbol at slot l is erased by user $\pi(i)$ in \hat{C}_π if and only if it is erased by *all* users $\pi(j)$ in channel C , with $j \leq i$, at slot l . This occurs with probability $\hat{\epsilon}_{\pi(i)} \triangleq \epsilon_{\cup_{j=1}^i \{\pi(j)\}}$. A straightforward calculation reveals that it holds $X \rightarrow Y_{\pi(N)} \rightarrow \dots \rightarrow Y_{\pi(1)}$. Hence, choosing the permutation $\hat{\pi}$ in Definition 1 such that $\hat{\pi}(i) = \pi(N - i + 1)$, we deduce that channel \hat{C}_π is physically degraded.

In fact, channel \hat{C}_π can be viewed as an augmentation of the original channel C , where additional error-free virtual channels are introduced between the receivers. Specifically, each user $\pi(i)$ in \hat{C}_π , for $1 \leq i \leq N-1$, sends its output symbol to user $\pi(i+1)$ through an error-free channel. Hence, any achievable rate for channel C can also be achieved for \hat{C}_π using the same code as in C and ignoring any symbols transmitted through the virtual channels. Denoting with \mathcal{C}_{FB} , $\hat{\mathcal{C}}_{\pi,FB}$ the feedback capacity regions of channels C , \hat{C}_π , respectively, we conclude that it holds $\mathcal{C}_{FB} \subseteq \hat{\mathcal{C}}_{\pi,FB}$.

The above set inclusion already provides an outer bound to \mathcal{C}_{FB} . In order to derive this bound, we note that the previous results imply that the feedback capacity region of the physically degraded channel \hat{C}_π is identical, due to Lemma 1, to the capacity region of \hat{C}_π without feedback. The latter is described, in general form, in Lemma 2 whence the following result follows.

Lemma 3: The feedback capacity region of \hat{C}_π is given by

$$\hat{\mathcal{C}}_{\pi,FB} = \left\{ \mathbf{R} \geq \mathbf{0} : \sum_{i \in \mathcal{N}} \frac{R_{\pi(i)}}{1 - \hat{\epsilon}_{\pi(i)}} \leq L \right\}. \quad (2)$$

The above analysis was based on a particular permutation π . Considering all $N!$ permutations on \mathcal{N} provides a tighter general outer bound.

Lemma 4: It holds $\mathcal{C}_{FB} \subseteq \mathcal{C}^{out} \triangleq \bigcap_{\pi \in \mathcal{P}} \hat{\mathcal{C}}_{\pi,FB}$, where \mathcal{P} is the set of all possible permutations on \mathcal{N} .

The outer bound \mathcal{C}^{out} has been derived based on the decoding rule in Section II, i.e. each user in channel C knows only its own feedback at each slot (hereafter referred to as “private” feedback). This raises a question regarding whether this bound is also valid for publicly available feedback (i.e. when each user in C knows the feedback from all other users at each slot). This question can be answered in the affirmative by extending the bounding arguments in the recent work of [21], which considered the case $N = 3$ and public feedback (which corresponds to a decoding function of the form $g_i(Y_i^n, \mathbf{Z}^n)$), to general N . Since the use of public feedback simplifies the presentation of the proposed algorithms, we initially assume that public feedback is available and propose a coding algorithm named `CODE1pub`. We remove this assumption later in Section V-A by proposing a simple overhead scheme on top of the former algorithm, which leads to a new algorithm, named `CODE1pri`, that only requires private feedback.

IV. A CLASS OF CODES

In this Section, we present a class of codes, collectively referred to as algorithm `CODE1pub` (the index emphasizes the assumption of public feedback), and describe the basic properties that guarantee its correctness.

A. The intuition behind the algorithm

Before the algorithm's description, a brief discussion of its underlying rationale will be useful. Since each user i must decode exactly the $|\mathcal{K}_i|$ packets in its session and a packet is an L -bit representation of an element in \mathbb{F}_q , the transmitter transmits appropriate linear combinations of packets so that each user i eventually receives $|\mathcal{K}_i|$ linearly independent combinations of the packets in \mathcal{K}_i . Hence, all quantities appearing in subsequent expressions are elements of \mathbb{F}_q and all linear operations are performed in \mathbb{F}_q .

The algorithm's operation can be summarized as follows: the transmitter maintains a set of virtual queues $Q_{\mathcal{S}}$, indexed by all non empty subsets $\mathcal{S} \subseteq \mathcal{N}$ and properly initialized, as well as queues Q_{D_i} for $i \in \mathcal{N}$. The queues Q_{D_i} contain copies of the packets that have been successfully received by user $i \in \mathcal{N}$. The algorithm processes each queue $Q_{\mathcal{S}}$ sequentially; during the processing of each queue $Q_{\mathcal{S}}$, the packet s to be transmitted next is selected as a linear combination of all packets currently stored in $Q_{\mathcal{S}}$, i.e. $s = \sum_{p \in Q_{\mathcal{S}}} a_s(p)p$, where $a_s(p)$ are suitably chosen coefficients in \mathbb{F}_q . Notice that, unless $a_s(p)$ is non-zero for exactly one $p \in Q_{\mathcal{S}}$, the transmitted packet s is not actually stored in $Q_{\mathcal{S}}$ but is created on-the-fly.

After transmitting s , the transmitter gets the ACK/NACKs for s from the receivers and (depending on which users successfully received s) potentially adds packet s into a single queue $Q_{\mathcal{S}'}$, with $\mathcal{S}' \supset \mathcal{S}$, and/or to queues Q_{D_i} , for all users i that received the packet. Some additional bookkeeping, to be described in detail in Section IV-B, is also performed. The algorithm terminates when all queues $Q_{\mathcal{S}}$ have been processed, at which point each user i can decode its original packets based on the packets contained in Q_{D_i} .

A central concept in the proposed algorithm is the notion of “token” which is defined as follows.

Definition 2: A packet s is a token for user i iff s can be written in the form

$$s = \sum_{p \in \mathcal{K}_i} b_s^{(i)}(p)p + c_s^{(i)}, \quad (3)$$

where $\mathbf{b}_s^{(i)} \triangleq (b_s^{(i)}(p), p \in \mathcal{K}_i)$, $c_s^{(i)} \in \mathbb{F}_q$ are known to user i . We call $\mathbf{b}_s^{(i)}$ the “coefficient vector” of packet s for user i .

In words, a token for user i is any packet s that allows i , upon reception of s , to effectively construct a linear equation with the packets in \mathcal{K}_i as unknowns (since $\mathbf{b}_s^{(i)}$, $c_s^{(i)}$ are known). For efficiency reasons, this equation should ideally be linearly independent w.r.t. all equations constructed by user i through the previously received packets (equivalently, $\mathbf{b}_s^{(i)} \notin \text{span}(\{\mathbf{b}_{s'}^{(i)} : s' \text{ received by } i \text{ prior to } s\})$). In this case, borrowing from network coding terminology, the packet is considered to be an “innovative” token.

Hence, each user i must receive $|\mathcal{K}_i|$ innovative tokens in order to decode its packets, at which point the algorithm stops. Notice that it is possible, and actually very desirable for throughput purposes, for a packet to simultaneously be a token (better yet, innovative token) for multiple users. In the context of this paper, we introduce the related, but not identical, notion of a “*Basis*” token, rigorously defined in Section IV-C, which is needed for the proof of the algorithm’s correctness and its performance analysis. However, we will still use the notion of “innovative” token to gain some intuition into the algorithm.

An important remark that follows from the previous discussion is that a token for user i may not only be non-innovative for i , but it may actually “contain” no packet intended for it, i.e. $b_s^{(i)}(p) = 0$ for all $p \in \mathcal{K}_i$. For example, consider the case where the transmitter sends a packet $s = p_1$, where $p_1 \in \mathcal{K}_1$ and s is received by user 2 only. Using the delta Kronecker $\delta_{m,n}$ notation and setting $\mathbf{b}_s^{(1)} = (\delta_{p,p_1} : p \in \mathcal{K}_1)$, $c_s^{(1)} = 0$, $\mathbf{b}_s^{(2)} = \mathbf{0}$ and $c_s^{(2)} = p_1$, it is easy to see that s is a token, according to Definition 2, for both users 1, 2 and none other. However, packet s contains no packet intended for 2, so that one could deduce that this slot was “wasted”. Of course, this is not actually the case (i.e. the slot was not really “wasted”) since user 2 gained some side information, so that the question now becomes how to optimally exploit the side information obtained through overhearing.

A distinctive characteristic of the proposed algorithm is that it efficiently exploits such cases (where users receive packets that are of no direct interest to them) by placing the packets into proper queues instead of discarding them. This results in better opportunities for efficient packet combinations in the future by creating simultaneous innovative tokens for multiple users and essentially compensating for previously “wasted” slots. The crux of the algorithm is in the careful bookkeeping required to handle these cases in an efficient manner and ensure that all users eventually receive the necessary number of innovative tokens.

The following proposition, which establishes that any linear combination of tokens is a new token (not necessarily innovative), will be useful.

Proposition 1: Consider a set of packets in a queue Q and a set of users \mathcal{S} such that each packet $p \in Q$ is a token for all users $i \in \mathcal{S}$. Then, any linear combination $s = \sum_{p \in Q} a_s(p)p$ of the packets in Q is a token for all $i \in \mathcal{S}$, provided that $a_s(p)$ are known to all users $i \in \mathcal{S}$.

The above proposition is easily proved by noting that each packet/token $p \in Q$ for user $i \in \mathcal{S}$ can be

Algorithm CODE1_{pub}

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1: initialize  $Q_S, K_S^i$  for all  $S \subseteq \mathcal{N}$  and  $i \in S$  and  $Q_{D_i}, K_{D_i}$  for all  $i \in \mathcal{N}$ ;
2:  $t \leftarrow 0$ ;
3: for  $\ell \leftarrow 1, \dots, N$  do
4:   for all ( $Q_S$  with  $|S| = \ell$  do) ▷ arbitrary order of processing
5:     while ( $K_S^i(t) > 0$  for at least one  $i \in S$ ) do
6:       compute suitable coefficients  $(a_s(p), p \in Q_S)$ ;
7:       transmit packet  $s = \sum_{p \in Q_S} a_s(p)p$ ;
8:       apply procedure ACTFB1 based on receiver feedback for  $s$ ;
9:        $t \leftarrow t + 1$ ;
10:    end while
11:  end for
12: end for

```

Fig. 1. Pseudocode for algorithm CODE1_{pub}.

written as $p = \sum_{u \in \mathcal{K}_i} b_p^{(i)}(u)u + c_p^{(i)}$, whence it follows

$$s = \sum_{u \in \mathcal{K}_i} \underbrace{\left(\sum_{p \in Q} a_s(p) b_p^{(i)}(u) \right)}_{b_s^{(i)}(u)} u + \underbrace{\left(\sum_{p \in Q} a_s(p) c_p^{(i)} \right)}_{c_s^{(i)}}, \quad (4)$$

so that s is still a token for each $i \in S$.

B. Description of algorithm CODE1_{pub}

Algorithm CODE1_{pub} is succinctly described in pseudocode form in Fig. 1. Specifically, the transmitter maintains a network of virtual queues Q_S , indexed by the non-empty subsets S of \mathcal{N} , as well as N queues denoted as Q_{D_i} , for $i \in \mathcal{N}$. Fig. 2 provides an illustration for 4 users, where an oval box represents Q_S , for the corresponding set S appearing as the box label, a square box represents queue Q_{D_i} and the vertical lines are used to classify the queues into “levels”, as will be explained below. The solid (dotted) line arrows indicate potential packet movement into a queue Q_S (Q_{D_i}). For graphical clarity, Fig. 2 only shows the packet movements originating from queues $Q_{\{1\}}, Q_{\{1,3\}}$; however, similar packet movements are allowed for the other queues, as will be explained soon.

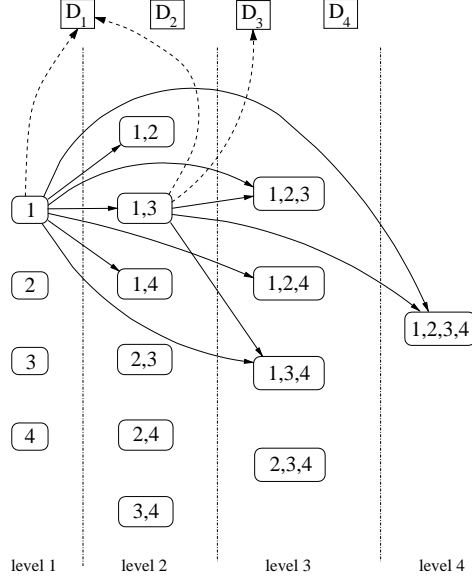


Fig. 2. Transmitter side virtual queue network for 4 users. Each oval box represents the queue indexed by the corresponding subset of $\{1, 2, 3, 4\}$, while solid and dotted line arrows indicate potential packet movement between the queues.

1) *Initialization:* All Q_{D_i} queues are initially empty while Q_S are initialized with the unicast packets as follows:

$$Q_S(0) = \begin{cases} \mathcal{K}_i & \text{if } S = \{i\}, \\ \emptyset & \text{otherwise.} \end{cases} \quad (5)$$

The performed initialization guarantees that all packets placed in queues $Q_{\{i\}}$ are tokens for user $i \in \mathcal{N}$ according to Definition 2.

Additionally, the algorithm keeps track of non-negative integer indices K_{D_i} , K_S^i . The former are associated to queues Q_{D_i} , for all $i \in \mathcal{N}$, while the latter are associated to queues Q_S , for all $S \subseteq \mathcal{N}$, $i \in S$. The indices K_{D_i} are initialized to 0 for all $i \in \mathcal{N}$, while K_S^i are initialized as

$$K_S^i(0) = \begin{cases} |\mathcal{K}_i| & \text{if } S = \{i\}, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

The entities Q_S , Q_{D_i} , K_S^i , K_{D_i} will be dynamically updated during the algorithm's execution (depending on the exact ACK/NACKs reported by the users), which is why we placed an explicit time dependence in (5), (6). In fact, the following note on notation will be useful: we write K_S^i to refer to the index when the exact instant at which the index is examined is unimportant (this is akin to using a variable name in a programming language: although the contents of the variable may change over time, we can always refer to the variable by name). We write $K_S^i(t)$ when we specifically refer to the value of the index at time

t . Furthermore, since the values of these indices depend on the erasures that occur, $K_{\mathcal{S}}^i(t)$ is actually a random variable. We will use the notation $\dot{K}_{\mathcal{S}}^i(t)$ (or $\dot{K}_{\mathcal{S}}^i$ when time is unimportant) when we want to emphasize the random nature of the indices.

For each user $i \in \mathcal{N}$, the algorithm also keeps track of subsets, denoted as $\mathcal{B}_{\mathcal{S}}^{(i)}$ (for $\mathcal{S} \subseteq \mathcal{N}$ with $i \in \mathcal{S}$, and \mathcal{B}_{D_i} , for $i \in \mathcal{N}$), of coefficient vectors $\mathbf{b}_s^{(i)}$ of tokens s for user i stored in $Q_{\mathcal{S}}$, Q_{D_i} (respectively). These coefficient vector sets, which will be seen to have the important property that they can be selected so that their union forms a basis for vector space $\mathbb{F}_q^{|\mathcal{K}_i|}$ for all $i \in \mathcal{N}$, are initialized as $\mathcal{B}_{D_i}(0) = \emptyset$ for $i \in \mathcal{N}$ and

$$\mathcal{B}_{\mathcal{S}}^{(i)}(0) = \begin{cases} \text{standard_basis}(\mathbb{F}_q^{|\mathcal{K}_i|}) & \text{if } \mathcal{S} = \{i\}, \\ \emptyset & \text{otherwise,} \end{cases} \quad (7)$$

where the *standard_basis* of an $|\mathcal{K}_i|$ -dimensional vector space is the set of vectors \mathbf{e}_i which have all components equal to zero except for the i -th component, which is set to one.

2) *Encoding*: We define as “level ℓ ” the groups of all queues $Q_{\mathcal{S}}$ with $|\mathcal{S}| = \ell$. The algorithm operates in N phases so that in phase ℓ , with $1 \leq \ell \leq N$, only transmissions of linear combinations of packets in one of the queues in level ℓ occur. Specifically, at phase ℓ , the transmitter orders the queues in level ℓ according to a predetermined rule, known to all users (say, according to lexicographic order of the index set \mathcal{S} , which corresponds to the top-to-bottom ordering shown in Fig. 2). The transmitter then examines the first, according to this order, queue $Q_{\mathcal{S}}$ and transmits a packet s that is a linear combination of all packets in $Q_{\mathcal{S}}$, i.e.

$$s = \sum_{p \in Q_{\mathcal{S}}} a_s(p)p. \quad (8)$$

We slightly abuse parlance and say that “ s is transmitted from $Q_{\mathcal{S}}$ ”, although it is clear that s is not actually stored in $Q_{\mathcal{S}}$ but is created on-the-fly. Proposition 1 guarantees that s is a token for all users $i \in \mathcal{S}$, provided that all packets $p \in Q_{\mathcal{S}}$ are also tokens for all $i \in \mathcal{S}$.

The exact generation method for $a_s(p)$ is unimportant as long as two general criteria are met.

Criterion 1: The procedure for generating $a_s(p)$ is known to all users, so that they can always reproduce the values of $a_s(p)$ even when they don’t receive the packet s . This implies that the receivers also know the size of all queues $Q_{\mathcal{S}}$, $\mathcal{S} \subseteq \mathcal{N}$, at all times.

Criterion 2: Assume that at the beginning of slot t , there exist (possibly empty) sets of vectors $\mathcal{B}_{D_i}(t) \subseteq \{\mathbf{b}_p^{(i)} : p \in Q_{D_i}(t)\}$, for all $i \in \mathcal{N}$, and $\mathcal{B}_{\mathcal{I}}^{(i)}(t) \subseteq \{\mathbf{b}_p^{(i)} : p \in Q_{\mathcal{I}}(t)\}$, for all $\mathcal{I} \subseteq \mathcal{N}$ and $i \in \mathcal{I}$, with the following properties (note that for $t = 0$, these properties automatically hold by

selecting $\mathcal{B}_{\mathcal{I}}^{(i)}(0)$, $\mathcal{B}_{D_i}(0)$ according to (7):

$$\begin{aligned} |\mathcal{B}_{\mathcal{I}}^{(i)}(t)| &= K_{\mathcal{I}}^i(t) \text{ and } |\mathcal{B}_{D_i}(t)| = K_{D_i}(t), \\ \mathcal{B}_{D_i}(t) \cup \bigcup_{\substack{\mathcal{I}:\mathcal{I}\subseteq\mathcal{N} \\ K_{\mathcal{I}}^i(t)>0}} \mathcal{B}_{\mathcal{I}}^{(i)}(t) &\text{ is a basis of } \mathbb{F}_q^{|\mathcal{K}_i|} \text{ for all } i \in \mathcal{N}, \end{aligned} \quad (9)$$

and, for each $i \in \mathcal{S}$ with $K_{\mathcal{S}}^i(t) > 0$, we pick an *arbitrary* $\hat{\mathbf{b}}_i \in \mathcal{B}_{\mathcal{S}}^{(i)}(t)$. Then, the generating algorithm for $a_s(p)$ should return as output any $(a_s(p) : p \in Q_{\mathcal{S}})$ such that the transmitted packet $s = \sum_{p \in Q_{\mathcal{S}}} a_s(p)p$ has a corresponding coefficient vector $\mathbf{b}_s^{(i)}$ with the property

$$\{\mathbf{b}_s^{(i)}\} \cup \mathcal{B}_{D_i}(t) \cup \bigcup_{\substack{\mathcal{I}:\mathcal{I}\subseteq\mathcal{N} \\ K_{\mathcal{I}}^i(t)>0}} \mathcal{B}_{\mathcal{I}}^{(i)}(t) - \{\hat{\mathbf{b}}_i\} \text{ is a basis of } \mathbb{F}_q^{|\mathcal{K}_i|} \text{ for all } i \in \mathcal{S} \text{ with } K_{\mathcal{S}}^i(t) > 0. \quad (10)$$

The above Criteria should be interpreted as two tests that any generating algorithm should pass, and conformance to these criteria is what the term “suitable coefficients” appearing in line 6 of Fig. 1 actually means. It is important to note that Criterion 2 is essentially a conditional result: it requires that the generator of $a_s(p)$ returns an output that satisfies (10) *provided* that there exist sets $\mathcal{B}_{\mathcal{I}}^{(i)}(t)$, $\mathcal{B}_{D_i}(t)$ that satisfy (9), without making any claims about the *actual* existence of these sets in the first place. It will be shown later (Lemmas 5, 6) that there actually exist sets $\mathcal{B}_{\mathcal{I}}^{(i)}(t)$, $\mathcal{B}_{D_i}(t)$ that satisfy (9) and, furthermore, there always exist $(a_s(p) : p \in Q_{\mathcal{S}})$ that satisfy (10) for $L > \log_2 N$.

Of the two Criteria, the second one is clearly the more difficult to satisfy. It will be shown that if coefficients $a_s(p)$ are selected so as to satisfy both Criteria, all users in \mathcal{N} will eventually receive a sufficient number of packets to individually solve a linear system that has a full rank matrix w.p. 1. Criterion 2 can be relaxed so that the generator of $a_s(p)$ returns output that satisfies (10) with probability arbitrarily close to 1; this choice leads to a simple generator for $a_s(p)$ based on random selection. Both variants of Criterion 2 can be satisfied by choosing a sufficiently large field size q ; however, for ease of presentation, we only consider the case where (10) is true w.p. 1.

3) *Feedback-based actions*: Once the linear combination s , in the form of (8), is transmitted from $Q_{\mathcal{S}}$ at slot t and the transmitter receives the corresponding feedback from all users, the following actions (or steps), collectively referred to as ACTFB1, are taken (all 4 cases must be examined, since they are not mutually exclusive). We denote with \mathcal{G} the set of users that successfully received s and omit the t dependence from all $K_{\mathcal{S}}^i$ indices.

ACTFB1 actions:

- 1) if no user in \mathcal{N} receives s , it is retransmitted.
- 2) if it holds $\mathcal{G} \subseteq \mathcal{S}$ and $K_{\mathcal{G}}^i = 0$ for all $i \in \mathcal{G}$, then s is retransmitted.

- 3) for each user $i \in \mathcal{S}$ that receives s and satisfies $K_{\mathcal{S}}^i > 0$, $K_{\mathcal{S}}^i$ is decreased by 1 and K_{D_i} is increased by 1.
- 4) if s has been erased by at least one user $i \in \mathcal{S}$ and it holds $\mathcal{G} \cap (\mathcal{N} - \mathcal{S}) \neq \emptyset$, then
 - packet s is added to queue $Q_{\mathcal{S} \cup \mathcal{G}}$.
 - for each user $i \in \mathcal{S}$ that erased s and satisfies $K_{\mathcal{S}}^i > 0$, $K_{\mathcal{S}}^i$ is decreased by 1 and $K_{\mathcal{S} \cup \mathcal{G}}^i$ is increased by 1.

No new coefficients are produced for the retransmissions in steps 1, 2. Fig. 2 presents the permissible token movements from queues $Q_{\{1\}}$, $Q_{\{1,3\}}$ that occur in step 4 of ACTFB1, where, for graphical clarity, transitions from the other queues are not shown (dashed lines correspond to step 3 of ACTFB1). Hence, a packet s transmitted from $Q_{\mathcal{S}}$ can only be moved to a queue $Q_{\mathcal{T}}$ with $\mathcal{T} \supset \mathcal{S}$ (a copy of the packet is also added to queue Q_{D_i} if s was correctly received by user i).

4) *Algorithm termination and decoding procedure:* Processing of $Q_{\mathcal{S}}$ (i.e. transmission of linear combinations of packets from $Q_{\mathcal{S}}$) continues for as long as there exists at least one $i \in \mathcal{S}$ with $K_{\mathcal{S}}^i > 0$. When it holds $K_{\mathcal{S}}^i = 0$ for all $i \in \mathcal{S}$, the transmitter moves to the next queue $Q_{\mathcal{S}'}$ in level ℓ and repeats the above procedure until it has processed all queues in level ℓ . When this occurs, phase ℓ is complete and the algorithm moves to phase $\ell + 1$, where it processes the queues in level $\ell + 1$.

Since the session length $\mathbf{K} = (|\mathcal{K}_i| : i \in \mathcal{N})$ and the exact algorithm for generating coefficients $a_s(p)$ are known to *all* users before execution of CODE1_{pub} begins, the presence of public feedback implies that, at the end of each slot, *all* users individually have exactly the same feedback information as the transmitter. Hence, they can “replay” the execution of CODE1_{pub} in real time and iteratively compute $\mathbf{b}_s^{(i)}$, c_s^i for each transmitted packet through (4) so that, by the time CODE1_{pub} terminates at the end of phase N , each user i has received sufficiently many tokens (i.e. packets stored in Q_{D_i}) to solve the related system of equations and decode the packets in \mathcal{K}_i .

C. Properties and correctness of CODE1_{pub}

The following two Lemmas, proved in Appendices A, B, respectively, contain all important properties of CODE1_{pub}, as they follow from its construction.

Lemma 5: During the execution of CODE1_{pub}, the following statements are true:

- 1) Any packet s that is stored in a queue $Q_{\mathcal{S}}$ at slot t , with $|\mathcal{S}| \geq 2$, is a linear combination of all packets in queue $Q_{\mathcal{I}_s}$ (for some non-empty set $\mathcal{I}_s \subset \mathcal{S}$) that has been transmitted at some prior slot $\tau < t$ and received (at slot τ) by all users in set $\mathcal{S} - \mathcal{I}_s$ and erased by all users in set $\mathcal{N} - \mathcal{S}$.

- 2) Any packet s stored in queue Q_S can be decomposed as $s = \sum_{u \in \cup_{j \in S} \mathcal{K}_j} \tilde{a}_s(u)u$, i.e. packet s is effectively a linear combination of packets destined for users in set S only.
- 3) Any packet s stored in Q_S is a token for all $i \in S$ (and only these $i \in S$).
- 4) When transmitting a linear combination s from Q_S at slot t , there always exist coefficients $a_s(p)$ that satisfy (10) of Criterion 2, provided that there exist sets that satisfy (9) and it holds $L > \log_2 N$.

The relation $L > \log_2 N$ will be assumed for the remainder of the paper, so that all subsequent results (Theorems, Lemmas etc) are based on this assumption. The following result essentially shows that each user $i \in \mathcal{N}$ is able to decode its packets by the end of CODE1_{pub}'s execution. The result is proved by induction, using the algorithm's initialization and the fourth statement in Lemma 5 to establish the crucial inductive step.

Lemma 6: Under the application of CODE1_{pub}, the following condition is true at the *beginning* of each slot t : there exist vector sets $\mathcal{B}_{\mathcal{I}}^{(i)}(t) \subseteq \{\mathbf{b}_p^{(i)} : p \in Q_{\mathcal{I}}(t)\}$, for all $\mathcal{I} \subseteq \mathcal{N}$ and $i \in \mathcal{I}$, and $\mathcal{B}_{D_i}(t) \subseteq \{\mathbf{b}_p^{(i)} : p \in Q_{D_i}(t)\}$, for all $i \in \mathcal{N}$, such that

- $|\mathcal{B}_{\mathcal{I}}^{(i)}(t)| = K_{\mathcal{I}}^i(t)$ and $|\mathcal{B}_{D_i}(t)| = K_{D_i}(t)$.
- $\mathcal{B}_{D_i}(t) \cup \bigcup_{\substack{\mathcal{I}: \mathcal{I} \subseteq \mathcal{N} \\ K_{\mathcal{I}}^i(t) > 0}} \mathcal{B}_{\mathcal{I}}^{(i)}(t)$ is a basis of $\mathbb{F}_q^{|\mathcal{K}_i|}$ for all $i \in \mathcal{N}$.

The existence of the above sets motivates the following definition.

Definition 3: A packet p is called a *Basis* token for user $i \in \mathcal{N}$ at slot t iff $\mathbf{b}_p^{(i)} \in \mathcal{B}_{D_i}(t) \cup \bigcup_{\substack{\mathcal{I}: \mathcal{I} \subseteq \mathcal{N} \\ K_{\mathcal{I}}^i(t) > 0}} \mathcal{B}_{\mathcal{I}}^{(i)}(t)$.

Clearly, at the beginning of the slot t_{end} immediately after the completion of phase N , Lemma 6 implies (since $K_{\mathcal{I}}^i(t_{end}) = 0$ for all i, \mathcal{I}) that $\mathcal{B}_{D_i}(t_{end})$ is a basis of $\mathbb{F}_q^{|\mathcal{K}_i|}$, for all $i \in \mathcal{N}$. Hence, each user i has received $|\mathcal{K}_i|$ linearly independent tokens (i.e. *Basis* tokens) and can decode its packets on a one-shot manner by solving the corresponding system of equations, using the *Basis* tokens in \mathcal{B}_{D_i} . Since this result holds for arbitrary channel statistics, CODE1_{pub} is, in principle, universally applicable. In addition, no prior knowledge of channel statistics is required for its execution.

D. Some further intuitive remarks

In retrospect, the combination of Lemmas 5, 6 and their methods of proof give a very intuitive explanation to the algorithm's operation, which we provide next. The sets $\mathcal{B}_{\mathcal{I}}^{(i)}(t)$ contain the vectors that span the *subspace* to which the $\mathbf{b}_s^{(i)}$ vector of any packet s received by user i from queue $Q_{\mathcal{I}}$ at slot t must belong in order to provide “useful” information to i (i.e. allow i to create an equation, w.r.t. packets in set \mathcal{K}_i , from the received s that is linearly independent w.r.t all previously created equations by user i). This follows from the fact that, for all $i \in \mathcal{N}$, any vector in $\text{span}(\mathcal{B}_{\mathcal{I}}^{(i)}(t))$ is linearly independent

w.r.t. the vectors in $\mathcal{B}_{D_i}(t)$ (i.e. the space spanned by the coefficient vectors of the tokens already received by user i), since the union of all these vector sets constitutes a basis of $\mathbb{F}_q^{|\mathcal{K}_i|}$. Similarly, $K_{\mathcal{I}}^i(t)$ is the number of the elements of the basis of $\mathbb{F}_q^{|\mathcal{K}_i|}$ that belong to $\mathcal{B}_{\mathcal{I}}^{(i)}(t)$.

Furthermore, by the algorithm's construction and Proposition 1, only the users $i \in \mathcal{S}$ can have *Basis* coefficient vectors corresponding to packets stored in $Q_{\mathcal{S}}$. This is due to Lemma 5, which states that any linear combination of packets in $Q_{\mathcal{S}}$ contains packets that are intended for users $i \in \mathcal{S}$ only. Similarly, Criterion 2 can be intuitively summarized as follows: when the algorithm processes queue $Q_{\mathcal{S}}$ and selects a packet $s = \sum_{p \in Q_{\mathcal{S}}} a_s(p)p$ for transmission at slot t , we should select $a_s(p)$ such that s is an innovative token for *all* $i \in \mathcal{S}$ with $K_{\mathcal{S}}^i(t) > 0$, *provided* that there exist certain sets with specific properties at slot t . The existence of these sets is guaranteed again by Lemma 6.

Regarding the rationale behind ACTFB1, step 3 of ACTFB1 is equivalent to saying that when user i receives a “useful” token at slot t (meaning that $K_{\mathcal{S}}^i(t) > 0$ so that there remain *Basis* tokens to receive) from $Q_{\mathcal{S}}$, this token should be added to \mathcal{B}_{D_i} (with a corresponding increase to K_{D_i}), so that it becomes a *Basis* token for user i at slot $t+1$. If this is not the case and there exist users, comprising set $\mathcal{G} \subseteq \mathcal{N} - \mathcal{S}$, who receive this packet (step 4 of ACTFB1), then the packet has become a token for users in $\mathcal{S} \cup \mathcal{G}$ and should be placed in queue $Q_{\mathcal{S} \cup \mathcal{G}}$. This allows the token to be simultaneously received by multiple users in the future and thus compensate for the current loss. Additionally, since user i can now recover this token more efficiently from $Q_{\mathcal{S} \cup \mathcal{G}}$ instead of $Q_{\mathcal{S}}$, the indices $K_{\mathcal{S}}^i$, $K_{\mathcal{S} \cup \mathcal{G}}^i$ should be modified accordingly to account for the token transition. Step 2 of ACTFB1 merely states that the packet is retransmitted when it is only received by users i who have already recovered from the queue all innovative tokens intended for them (i.e. $\mathcal{B}_{\mathcal{S}}^{(i)}$ is empty).

If $K_{\mathcal{S}}^i$ becomes 0 at the end of some slot \tilde{t} , queue $Q_{\mathcal{S}}$ is no longer useful for user i , since all linearly independent combinations that could be created from $Q_{\mathcal{S}}$ have either been received by i or stored in higher level queues (due to step 4 of ACTFB1) for future recovery by i . Of course, the queue is still useful for any other users $j \in \mathcal{S}$ with $K_{\mathcal{S}}^j(\tilde{t}) > 0$.

E. An example of execution of CODE1_{pub}

We next provide a concrete example of execution for CODE1_{pub} that illustrates some of the points mentioned in Sections IV-B, IV-C. We consider the case of 3 users with 10 packets destined to each of them and stored at the transmitter. We denote the sets of packets destined for user 1, 2, 3 as $\mathcal{K}_1 = \{u_1, \dots, u_{10}\}$, $\mathcal{K}_2 = \{v_1, \dots, v_{10}\}$, $\mathcal{K}_3 = \{w_1, \dots, w_{10}\}$, respectively. We also introduce an upper index notation to denote the set of users that have received a packet, e.g. $u_1^{(23)}$ denotes that packet u_1 was

received by users 2, 3 only.

The initialization of CODE1_{pub} is trivial: all packets of set \mathcal{K}_i are placed in queue $Q_{\{i\}}$, the indices are initialized as $T_{\{1\}}^1(0) = T_{\{2\}}^2(0) = T_{\{3\}}^3(0) = 10$ (all other indices are zero) and the basis sets are initialized as $\mathcal{B}_{\{1\}}^{(1)}(0) = \mathcal{B}_{\{2\}}^{(2)}(0) = \mathcal{B}_{\{3\}}^{(3)}(0) = \text{standard_basis}(\mathbb{F}^{10})$ (all other sets are empty). We denote with e_i the standard basis vector which has its i -th component set to 1.

CODE1_{pub} executes Phase 1, in which the queues $Q_{\{1\}}$, $Q_{\{2\}}$, $Q_{\{3\}}$ are sequentially processed in this order. The random erasure events that occur in each slot are shown in Table I, where R/E stands for Received/Erased, respectively, and X denotes an unimportant value (i.e. X can be either R or E but, in any case, does not affect the algorithm's actions). For example, the ERE for slot 2 of $Q_{\{1\}}$ denotes a transmission that was received only by user 2. We also use the following conventions in Table I:

- for simplicity, we omit any slots in which the packet must be retransmitted due to steps 1, 2 of ACTFB1. Hence, the slot number (1,2, etc) should not be interpreted as physical time but rather as an ordinal indicating slots in which no retransmission was required. In other words, slots 1, 2 need not be contiguous in time.
- due to the imposed order of processing, queues $Q_{\{2\}}$, $Q_{\{3\}}$ are actually processed in slots 11–20 and 21–30, respectively. The reader should interpret the rows corresponding to $Q_{\{2\}}$, $Q_{\{3\}}$ accordingly.

TABLE I
CODE1_{pub} EXECUTION. ERASURES AND QUEUE CONTENTS AT END OF PHASE 1.

Phase 1 execution										
Slot	1	2	3	4	5	6	7	8	9	10
Q_1	RXX	ERE	ERR	ERE	EER	ERR	RXX	EER	RXX	ERR
Q_2	REE	EER	XXR	REE	EER	XXR	RER	RER	XXR	EER
Q_3	ERE	REE	REE	XXR	ERE	XXR	ERE	ERE	REE	RRE
Queue status at end of phase 1										
Packets decoded by users	user 1: $u_1^{(1)}, u_7^{(1)}, u_9^{(1)}$, user 2: $v_3^{(2)}, v_6^{(2)}, v_9^{(2)}$, user 3: $w_4^{(3)}, w_6^{(3)}$									
$Q_{\{1,2\}}$ contents	$u_2^{(2)}, u_4^{(2)}, v_1^{(1)}, v_4^{(1)}$									
$Q_{\{1,3\}}$ contents	$u_5^{(3)}, u_8^{(3)}, w_2^{(1)}, w_3^{(1)}, w_9^{(1)}$									
$Q_{\{2,3\}}$ contents	$v_2^{(3)}, v_5^{(3)}, v_{10}^{(3)}, w_1^{(2)}, w_5^{(2)}, w_7^{(2)}, w_8^{(2)}$									
$Q_{\{1,2,3\}}$	$u_3^{(23)}, u_6^{(23)}, u_{10}^{(23)}, v_7^{(13)}, v_8^{(13)}, w_{10}^{(12)}$									
Basis sets at end of phase 1	$\mathcal{B}_{\{1,2\}}^{(1)} = \{e_2, e_4\}, \mathcal{B}_{\{1,2\}}^{(2)} = \{e_1, e_4\}, \mathcal{B}_{\{1,3\}}^{(1)} = \{e_5, e_8\}, \mathcal{B}_{\{1,3\}}^{(3)} = \{e_2, e_3, e_9\}$ $\mathcal{B}_{\{2,3\}}^{(2)} = \{e_2, e_5, e_{10}\}, \mathcal{B}_{\{2,3\}}^{(3)} = \{e_1, e_5, e_7, e_8\}, \mathcal{B}_{\{1,2,3\}}^{(1)} = \{e_3, e_6, e_{10}\}$ $\mathcal{B}_{\{1,2,3\}}^{(2)} = \{e_7, e_8\}, \mathcal{B}_{\{1,2,3\}}^{(3)} = \{e_{10}\}$ $\mathcal{B}_{D_1} = \{e_1, e_7, e_9\}, \mathcal{B}_{D_2} = \{e_3, e_6, e_9\}, \mathcal{B}_{D_3} = \{e_4, e_6\}$									

The transmitter starts processing $Q_{\{1\}}$ and sends the uncoded packet u_j in slot j . Similarly, when queues $Q_{\{2\}}, Q_{\{3\}}$ are processed, packet v_j, w_j is transmitted, respectively, in slot j . This packet selection policy complies with Criterion 2. Specifically, if, at slot j , the packet u_j is received by user 1, then its corresponding $\mathbf{b}_j^{(1)}$ vector (i.e. \mathbf{e}_j) is removed from set $\mathcal{B}_{\{1\}}^{(1)}$ and added to \mathcal{B}_{D_1} . If u_j is erased by user 1 and received by all users in set \mathcal{S} , then vector \mathbf{e}_j is moved from $\mathcal{B}_{\{1\}}^{(1)}$ to $\mathcal{B}_{\mathcal{S}}^{(1)}$. Similar actions are taken for packets v_j, w_j .

The queue contents at the end of phase 1 are also shown in Table I. Some packets have already been decoded by their respective destinations, while the rest have been distributed among the virtual queues. The indices at the end of phase 1 are as follows: $T_{\{1,2\}}^1 = T_{\{1,2\}}^2 = 2$, $T_{\{1,3\}}^1 = 2$, $T_{\{1,3\}}^3 = 3$, $T_{\{2,3\}}^2 = 3$, $T_{\{2,3\}}^3 = 4$, $T_{\{1,2,3\}}^1 = 3$, $T_{\{1,2,3\}}^2 = 2$ and $T_{\{1,2,3\}}^3 = 1$. For the sets $\mathcal{B}_{\mathcal{S}}^{(i)}$, the policy of sending uncoded packets in phase 1, combined with ACTFB1, implies that $\mathcal{B}_{\mathcal{S}}^{(i)}(t_1)$ (where t_1 denotes the end of phase 1) contains the unit basis vectors corresponding to the packets stored in $Q_{\mathcal{S}}$ at the end of the phase.

The algorithm now executes phase 2, in which the queues $Q_{\{1,2\}}, Q_{\{1,3\}}, Q_{\{2,3\}}$ are sequentially processed in this order. Criterion 2 cannot be satisfied by sending uncoded packets only, so the transmitter selects a proper linear combination of all packets in the queue currently being processed. Hence, the packet s_n transmitted at slot n of phase 2 has the form $s_n = \sum_{p \in Q_{\mathcal{S}}} a_{s_n}(p)p$, where $Q_{\mathcal{S}}$ is the queue being processed at slot n and $a_s(p)$ satisfy Criterion 2. The erasures that occur in phase 2 and the corresponding ACTFB1 actions, as well as their results, are shown in Table II (again, the slot number should be interpreted as ordinal instead of actual time).

The first 6 slots of phase 2 illustrate some of the finer points of the algorithm. Specifically, in slot 1 of phase 2, the transmitted packet s_1 is only received by user 1. Since the vector $\mathbf{b}_{s_1}^{(1)}$, corresponding to packet s_1 , belongs to the span of the vectors $\{\mathbf{b}_p^{(1)} : p \in Q_{\{1,2\}}\}$, it follows that $\mathbf{b}_{s_1}^{(1)} \in \text{span}(\mathcal{B}_{\{1,2\}}^{(1)}(t_1))$, i.e. $\mathbf{b}_{s_1}^{(1)} \in \text{span}(\mathbf{e}_2, \mathbf{e}_4)$. The packets received by user 1 up to now span the space $\text{span}(\mathcal{B}_{D_1}(t_1)) = \text{span}(\{\mathbf{e}_1, \mathbf{e}_7, \mathbf{e}_9\})$, so that s_1 brings innovative information for this user. Hence, $T_{\{1,2\}}^1$, which counts the number of innovative tokens that user 1 has yet to recover from $Q_{\{1,2\}}$, must be decreased by one.

In slot 2, the transmitted packet s_2 is received by users 2, 3. Using a similar argument as for user 1 in slot 1, we conclude that user 2 gains an innovative token (since $\mathbf{b}_{s_2}^{(2)} \in \text{span}(\{\mathbf{b}_p^{(2)} : p \in Q_{\{1,2\}}\}) = \text{span}(\mathbf{e}_1, \mathbf{e}_4)$ and $\mathcal{B}_{D_2} = \{\mathbf{e}_3, \mathbf{e}_6, \mathbf{e}_9\}$) and the $T_{\{1,2\}}^2$ index must be accordingly reduced. It is important to note that, since at the time of transmission of s_2 it holds $T_{\{1,2\}}^1 > 0$, s_2 is also an innovative token for user 1. Additionally, s_2 is a token for users 1, 2 (due to Lemma 5) and 3 (since user 3 received s_2), so it is moved to queue $Q_{\{1,2,3\}}$. Hence, user 1 can now recover the innovative token corresponding to packet s_2 from queue $Q_{\{1,2,3\}}$ instead of $Q_{\{1,2\}}$, so that the $T_{\{1,2\}}^1, T_{\{1,2,3\}}^1$ indices are modified accordingly.

TABLE II
CODE1_{pub} EXECUTION. ERASURES AND QUEUE CONTENTS AT END OF PHASE 2.

Phase 2										
Processing	$Q_{\{1,2\}}$			$Q_{\{1,3\}}$			$Q_{\{2,3\}}$			
Slot	1	2	3	4	5	6	7	8	9	10
Erasure event	REE	ERR	RER	ERE	ERE	ERE	REE	ERR	ERR	EER
Applicable ACTFB1 actions	3	3,4	3,4	4	4	4	4	3	3	3
	$T_{\{1,2\}}^1, T_{\{1,2\}}^2$			$T_{\{1,3\}}^1, T_{\{1,3\}}^3$			$T_{\{2,3\}}^2, T_{\{2,3\}}^3$			
Index value at end of slot	1,2	0,1	0,0	1,2	0,1	0,0	2,3	1,2	0,1	0,0
Queue contents at end of phase 2										
$Q_{\{1,2,3\}}$	$u_3^{(23)}, u_6^{(23)}, u_{10}^{(23)}, v_7^{(13)}, v_8^{(13)}, w_{10}^{(12)}, s_2^{(23)}, s_3^{(13)}, s_4^{(2)}, s_5^{(2)}, s_6^{(2)}, s_7^{(1)}$									
Basis sets for $Q_{\{1,2,3\}}$ at end of phase 2	$\mathcal{B}_{\{1,2,3\}}^{(1)} = \{e_3, e_6, e_{10}, b_{s_2}^{(1)} \in \text{span}(e_2, e_4), b_{s_4}^{(1)}, b_{s_5}^{(1)} \in \text{span}(e_5, e_8)\}$ $\mathcal{B}_{\{1,2,3\}}^{(2)} = \{e_7, e_8, b_{s_3}^{(2)} \in \text{span}(e_1, e_4), b_{s_7}^{(2)} \in \text{span}(e_2, e_5, e_{10})\}$ $\mathcal{B}_{\{1,2,3\}}^{(3)} = \{e_{10}, b_{s_4}^{(3)}, b_{s_5}^{(3)}, b_{s_6}^{(3)} \in \text{span}(e_2, e_3, e_9), b_{s_7}^{(3)} \in \text{span}(e_1, e_5, e_7, e_8)\}$									
Packets received by 1	$u_1, u_7, u_9, s_1, s_3, s_7$									
$b^{(1)}$ received by 1	$e_1, e_7, e_9, b_{s_1}^{(1)} \in \text{span}(e_2, e_4), b_{s_3}^{(1)} \in \text{span}(e_2, e_4), b_{s_7}^{(1)} = \mathbf{0}$									
Packets received by 2	$v_3, v_6, v_9, s_2, s_4, s_5, s_6, s_8, s_9$									
$b^{(2)}$ received by 2	$e_3, e_6, e_9, b_{s_2}^{(2)} \in \text{span}(e_1, e_4), b_{s_4}^{(2)} = b_{s_5}^{(2)} = b_{s_6}^{(2)} = \mathbf{0}$ $b_{s_8}^{(2)}, b_{s_9}^{(2)} \in \text{span}(e_2, e_5, e_{10})$									
Packets received by 3	$w_4, w_6, s_2, s_3, s_8, s_9, s_{10}$									
$b^{(3)}$ received by 3	$e_4, e_6, b_{s_2}^{(3)} = b_{s_3}^{(3)} = \mathbf{0}, b_{s_8}^{(3)}, b_{s_9}^{(3)}, b_{s_{10}}^{(3)} \in \text{span}(e_1, e_5, e_7, e_8)$									

Notice that, though s_2 becomes a token for user 3, it is *not* innovative for user 3 since it holds $b_{s_2}^{(3)} = \mathbf{0}$. A similar interpretation can be given for the actions in slot 3 by swapping the roles of users 1, 2.

In slots 4, 5, 6, the transmitted packets are only received by user 2, so that step 4 of ACTFB1 is applicable and all 3 transmitted packets are moved to $Q_{\{1,2,3\}}$. By construction of the algorithm, it also holds $b_{s_4}^{(1)}, b_{s_5}^{(1)}, b_{s_6}^{(1)} \in \text{span}(\mathcal{B}_{\{1,3\}}^{(1)}(t_1))$. Since, at the beginning of slot 4, the vectors in $\mathcal{B}_{\{1,3\}}^{(1)}$ span a subspace of dimension $T_{\{1,3\}}^1 = 2$ (due to Lemma 6), it follows that $b_{s_4}^{(1)}, b_{s_5}^{(1)}, b_{s_6}^{(1)}$ are linearly *dependent even though* $s_4, s_5, s_6 \in Q_{\{1,2,3\}}$. The last statement clearly demonstrates the true meaning of sets $\mathcal{B}_{\mathcal{S}}^{(i)}(t)$: these sets contain the vectors b corresponding to tokens that remain to be received by user i from queue $Q_{\mathcal{S}}$ at slot t . It is exactly due to the fact that the packets stored in $Q_{\mathcal{S}}$ are not simultaneously innovative for all users $i \in \mathcal{S}$ that the sets $\mathcal{B}_{\mathcal{S}}^{(i)}$ must be introduced in the first place.

At the end of phase 2 (denote this time as t_2), the indices for $Q_{\{1,2,3\}}$ are as follows: $T_{\{1,2,3\}}^1 = 6$, $T_{\{1,2,3\}}^2 = 4$, $T_{\{1,2,3\}}^3 = 5$. In phase 3, the transmitter sends linear combinations of all packets stored in

TABLE III
CODE1_{pub} EXECUTION. ERASURES AND QUEUE CONTENTS AT END OF PHASE 3.

Phase 3							
Processing	$Q_{\{1,2,3\}}$						
Slot	11	12	13	14	15	16	17
Erasure event	RRR	RER	RRR	ERR	RRE	RER	RRE
	$T_{\{1,2,3\}}^1, T_{\{1,2,3\}}^2, T_{\{1,2,3\}}^3$						
Index value at end of slot	5,3,4	4,3,3	3,2,2	3,1,1	2,0,1	1,0,0	0,0,0
Queue contents at end of phase 3							
Packets received by 1	$u_1, u_7, u_9, s_1, s_3, s_7, s_{11}, s_{12}, s_{13}, s_{15}, s_{16}, s_{17}$						
$b^{(1)}$ received by 1	$e_1, e_7, e_9, b_{s_1}^{(1)}, b_{s_3}^{(1)} \in \text{span}(e_2, e_4), b_{s_7}^{(1)} = \mathbf{0}$ $b_{s_{11}}^{(1)}, b_{s_{12}}^{(1)}, b_{s_{13}}^{(1)}, b_{s_{15}}^{(1)}, b_{s_{16}}^{(1)}, b_{s_{17}}^{(1)} \in \text{span}(e_3, e_1, e_{10}, b_{s_2}^{(1)}, b_{s_4}^{(1)}, b_{s_{10}}^{(1)})$						
Packets received by 2	$v_3, v_6, v_9, s_2, s_4, s_5, s_6, s_8, s_9, s_{11}, s_{13}, s_{14}, s_{15}, s_{17}$						
$b^{(2)}$ received by 2	$e_3, e_6, e_9, b_{s_2}^{(2)} \in \text{span}(e_1, e_4), b_{s_8}^{(2)}, b_{s_9}^{(2)} \in \text{span}(e_2, e_5, e_{10})$ $b_{s_{11}}^{(2)}, b_{s_{13}}^{(2)}, b_{s_{14}}^{(2)}, b_{s_{15}}^{(2)}, b_{s_{17}}^{(2)} \in \text{span}(e_7, e_8, b_{s_3}^{(2)}, b_{s_7}^{(2)})$						
Packets received by 3	$w_4, w_6, s_2, s_3, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{16}$						
$b^{(3)}$ received by 3	$e_4, e_6, b_{s_8}^{(3)}, b_{s_9}^{(3)}, b_{s_{10}}^{(3)}$ $b_{s_{11}}^{(3)}, b_{s_{12}}^{(3)}, b_{s_{13}}^{(3)}, b_{s_{14}}^{(3)}, b_{s_{16}}^{(3)} \in \text{span}(e_{10}, b_{s_4}^{(3)}, b_{s_5}^{(3)}, b_{s_6}^{(3)}, b_{s_7}^{(3)})$						

$Q_{\{1,2,3\}}$ until all T indices become zero. Table III shows the erasures that occurred and queue contents at the end of phase 3 (note that only step 3 of ACTFB1 is now applicable and the slot numbering in phase 3 continues from where phase 2 stopped). At the end of phase 3, each user has collected 10 innovative tokens (i.e. linearly independent equations) and can decode its packets by solving a linear system.

V. PERFORMANCE ANALYSIS FOR CODE1_{pub}

In this Section, we analyze the performance of CODE1_{pub} for arbitrary channel statistics and conclude that CODE1_{pub} achieves the capacity outer bound of Lemma 4 (i.e. achieves capacity), provided that the users in \mathcal{N} can be ordered according to a specific relation that depends on channel statistics and the chosen rates; this provision is shown to be true for the special case of symmetric channels, i.e. channels which satisfy the condition $\epsilon_{\mathcal{I}} = \epsilon_{\mathcal{J}}$, for all \mathcal{I}, \mathcal{J} with $|\mathcal{I}| = |\mathcal{J}|$ (i.e. the probability that all users in set \mathcal{I} erase a packet is a function of $|\mathcal{I}|$ only).

We also consider the case of spatially independent channels (i.e. $\epsilon_{\mathcal{I}} = \prod_{i \in \mathcal{I}} \epsilon_i$) with (one-sided) fairness constraints, a notion first introduced in [14]. To define this notion, we assume, without loss of generality, that it holds $\epsilon_1 \geq \dots \geq \epsilon_N$ and define a rate \mathbf{R} to be (one-sided) fair iff it belongs to the

set $\mathcal{R}_{fair} \triangleq \{(R_1, \dots, R_N) \geq \mathbf{0} : \epsilon_1 R_1 \geq \dots \geq \epsilon_N R_N\}$. We will subsequently show that CODE1_{pub} achieves any rate $\mathbf{R} \in \mathcal{C}^{out} \cap \mathcal{R}_{fair}$, i.e. CODE1_{pub} achieves all achievable fair rates for the BPEC channel.

The complete performance analysis for CODE1_{pub} is quite lengthy so, for the reader's convenience, we present here the main results.

Theorem 1: Denote $\hat{f}_S^i \triangleq \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i\}} \frac{(-1)^{|\mathcal{S}| - |\mathcal{H}| - 1}}{1 - \epsilon_{\mathcal{N} - \mathcal{H}}}$ for all $\mathcal{S} \subseteq \mathcal{N}$ with $i \in \mathcal{S}$. For arbitrary channel statistics, the rate region of CODE1_{pub} , in information bits per transmitted symbol, is given by

$$\mathcal{R}_{\text{CODE1}_{pub}} = \left\{ \mathbf{R} \geq \mathbf{0} : \sum_{\mathcal{S} \subseteq \mathcal{N}} \max_{i \in \mathcal{S}} (\hat{f}_S^i R_i) \leq L \right\}. \quad (11)$$

Outline of proof: We provide here an outline of the proof with complete details given in Appendix C. Since CODE1_{pub} , as described in Section IV-B, is a variable-length coding scheme (i.e. the total number of transmissions \dot{T}^* required by the algorithm is a random variable, hence unknown a priori), we propose the following modification to make it compatible with a fixed blocklength coding scheme that is required by the information-theoretic rate definition of Section II. For a given rate vector \mathbf{R} and fixed n , we create, for each user $i \in \mathcal{N}$, a set of packets \mathcal{K}_i , where $|\mathcal{K}_i| = K_i(\mathbf{R}) = \lceil nR_i \rceil$, and consider \mathcal{K}_i as the intended message for user i . We then apply CODE1_{pub} but stop at n transmissions and declare an error if CODE1_{pub} has not terminated yet (i.e. an error is declared if $\dot{T}^* > n$).

Hence, the modified fixed blocklength code has a probability of error $p_n(e) = \Pr(\dot{T}^* > n) = \Pr(\dot{T}^*/n > 1)$; furthermore, using the SLLN, we can show that \dot{T}^*/n tends to a deterministic quantity $\bar{T}^*(\mathbf{R})$ (the \mathbf{R} dependence is due to the fact that \dot{T}^* implicitly depends on $\mathbf{K} \triangleq \lceil n\mathbf{R} \rceil$) w.p. 1 as $n \rightarrow \infty$. Hence, the information-theoretic rate region achieved by CODE1_{pub} is the set of rates \mathbf{R} , measured in information symbols per transmission, for which $p_n(e) \rightarrow 0$ as $n \rightarrow \infty$, which is intuitively equal to $\{\mathbf{R} : \bar{T}^*(\mathbf{R}) \leq 1\}$. To compute the rate region in information bits per transmission, we use the fact that each symbol contains L bits and $\bar{T}^*(\mathbf{R})$ is a homogeneous function of degree 1 with respect to its argument (i.e. $\bar{T}^*(\alpha\mathbf{R}) = \alpha\bar{T}^*(\mathbf{R})$ for any $\alpha > 0$). Appendix C provides a detailed calculation of $\bar{T}^*(\mathbf{R})$ and makes the above argument rigorous. ■

In order to provide a general optimality criterion for CODE1_{pub} , we need to define the following set.

$$\mathcal{R}_{ord} \triangleq \left\{ \mathbf{R} \geq \mathbf{0} : \exists \text{ permutation } \tilde{\pi} \text{ s.t. } \forall \mathcal{S} \subseteq \mathcal{N} \text{ it holds } \arg \max_{i \in \mathcal{S}} (\hat{f}_S^i R_i) = \arg \min_{i \in \mathcal{S}} (\tilde{\pi}(i)) \right\}. \quad (12)$$

Although the permutation $\tilde{\pi}$ in (12) may implicitly depend on \mathbf{R} (as well as on channel statistics through \hat{f}_S^i) and should actually be written as $\tilde{\pi}_{\mathbf{R}}$, we opt to simplify the notation by henceforth omitting this dependence. In words, \mathcal{R}_{ord} contains all rates \mathbf{R} , whose indices can be rearranged according to $\tilde{\pi}$ so that

the relation in (12) is satisfied. Notice that \mathcal{R}_{ord} is a cone set, i.e. $\mathbf{R} \in \mathcal{R}_{ord}$ implies $\alpha \mathbf{R} \in \mathcal{R}_{ord}$ for all $\alpha \geq 0$. Hence, as long as there exists some non-zero $\mathbf{R} \in \mathcal{R}_{ord}$, the set \mathcal{R}_{ord} intersects the boundary of \mathcal{C}^{out} .

Introducing the subset of \mathcal{R}_{ord}

$$\mathcal{D} \triangleq \left\{ \mathbf{R} \in \mathcal{R}_{ord} : \sum_{i=1}^N \frac{R_{\tilde{\pi}^{-1}(i)}}{1 - \epsilon_{\{\tilde{\pi}^{-1}(1), \dots, \tilde{\pi}^{-1}(i)\}}} \leq L \right\}, \quad (13)$$

where $\tilde{\pi}$ is the permutation corresponding to $\mathbf{R} \in \mathcal{R}_{ord}$ via (12), we prove the following result in Appendix D.

Lemma 7: If $\mathbf{R} \in \mathcal{R}_{ord}$, it holds

$$\sum_{S \subseteq \mathcal{N}} \max_{i \in S} \left(\hat{f}_S^i R_i \right) = \sum_{i=1}^N \frac{R_{\tilde{\pi}^{-1}(i)}}{1 - \epsilon_{\{\tilde{\pi}^{-1}(1), \dots, \tilde{\pi}^{-1}(i)\}}}. \quad (14)$$

This implies, through Theorem 1, that $\mathcal{R}_{CODE1_{pub}} \cap \mathcal{R}_{ord} = \mathcal{D}$.

Theorem 1 and Lemma 7 now lead to the main optimality criterion.

Theorem 2: The rate region of CODE1_{pub} satisfies the relation $\mathcal{R}_{CODE1_{pub}} \cap \mathcal{R}_{ord} = \mathcal{C}^{out} \cap \mathcal{R}_{ord} = \mathcal{D}$ (i.e. CODE1_{pub} achieves any achievable rate in \mathcal{R}_{ord}). Therefore, if it holds $\mathcal{R}_{ord} \supseteq \mathcal{C}^{out}$, the rate region of CODE1_{pub} satisfies the relation $\mathcal{R}_{CODE1_{pub}} = \mathcal{C}^{out} = \mathcal{D}$, i.e. CODE1_{pub} achieves capacity.

More details are provided in Appendices C, D. Theorem 2 implies the following result (whose proof is given in Appendix E) regarding the optimality of CODE1_{pub} .

Theorem 3: The set \mathcal{R}_{ord} satisfies the following relations: 1) $\mathcal{R}_{ord} = \{\mathbf{R} : \mathbf{R} \geq \mathbf{0}\}$, for symmetric channels and 2) $\mathcal{R}_{ord} \supseteq \mathcal{R}_{fair}$ for spatially independent one-sided fair channels which satisfy the condition $\epsilon_1 \geq \dots \geq \epsilon_N$. Hence, CODE1_{pub} achieves capacity for symmetric channels and also achieves all rates in $\mathcal{R}_{fair} \cap \mathcal{C}^{out}$ for spatially independent channels.

A. Incorporation of overhead

The previous analysis rests on two assumptions: 1) *public* feedback is instantaneously available to *all* users, and 2) each user $i \in \mathcal{N}$ always knows the values of $\mathbf{b}_s^{(i)}$, $c_s^{(i)}$ for any packet s it receives. In order to remove the former assumption (so that each user need only know its own feedback), and still satisfy the latter requirement, the feedback information must be conveyed to the receivers by the transmitter at the expense of achievable rate (i.e. incorporation of overhead). In fact, the second requirement is equivalent to the requirement that all users know the coefficients $a_s(p)$ of *any* generated packet s , even if they don't receive it. This follows from the fact that all $\mathbf{b}_s^{(i)}$ are iteratively computed, through (4), based on the selected $a_s(p)$. Hence, the second requirement is satisfied if the algorithm for generating $a_s(p)$

(see the final remarks in the proof of Lemma 5) is available at each receiver. This eliminates the need for appending the coefficient vector into the packet header as was originally proposed in [22]. We next describe a simple, not necessarily optimal, overhead scheme that can be applied to the original algorithm CODE1_{pub} (with or without the fixed blocklength modification) and leads to a new algorithm, named CODE1_{pri} , which does not require public feedback. The latter algorithm consists of two stages, called “pure information transmission” and “feedback recovery”, as is explained next.

During the pure information transmission stage, a single overhead bit h_1 is reserved in each packet of length L . Hence, the information payload contains $L - 1$ bits and the linear combinations are performed only over the information payload (i.e. we treat the sequence of $L - 1$ bits as an element of \mathbb{F}_q). The transmitter executes CODE1_{pub} normally¹, by setting $h_1 = 0$ in each transmitted packet and taking the received feedback into account according to ACTFB1. For each transmitted linear combination s (including retransmissions due to steps 1, 2 of ACTFB1), the transmitter also creates an N -bit group (f_1, \dots, f_N) , where f_i is 1 or 0, depending on whether or not user i received s , and stores it into a feedback log. Denoting with \dot{T}^* the (random) number of time slots required by CODE1_{pub} to process all queues, an equal number of N -bit groups is created and added to the feedback log. Meanwhile, each user stores the packets it receives in a single queue in a FIFO manner since, at this point, it can do nothing more without additional information on the other users’ feedback.

In principle, if each user learns the exact feedback log, it will gain the same information it would have in the case of public feedback; hence, it can “replay” the algorithm as it was executed at the transmitter side and deduce the values of $\mathbf{b}_s^{(i)}$, $c_s^{(i)}$ for the packets s it received. Hence, the objective now becomes to multicast the feedback log to all N users in a manner that does not introduce significant overhead. This is performed in the second stage of feedback recovery, in which 2 overhead bits h_1, h_2 are reserved for each packet. When CODE1_{pub} terminates (i.e. phase N is complete), the transmitter splits the entire feedback log into packets of length L (so that a total of $\lceil N\dot{T}^*/(L - 2) \rceil$ packets is required, considering the 2 bit overhead per packet; we hereafter call these “feedback” packets) and broadcasts each feedback

¹based on the algorithm’s description in Section IV-B, the reader will notice that the existence of public feedback may affect the exact decoding procedure at each user but does *not* affect the transmitter’s actions in any way, since the latter always has access to feedback from *all* users.

packet until it is received by all users.² Notice that a single feedback packet actually contains the exact feedback that occurred in a group of $\lfloor (L - 2)/N \rfloor$ consecutive slots.

Each feedback packet has its h_1 bit set to 1, so any user that receives it can distinguish it from “pure” information packets (which had $h_1 = 0$) received during the previous phases of CODE1_{pub} . Furthermore, the transmitter applies the following procedure for bit h_2 . The first transmitted feedback packet has $h_2 = 0$. The transmitter keeps sending this packet until *all* users receive it. When this occurs, the transmitter sends the next feedback packet by flipping the h_2 bit.

The flipping of the h_2 bit is necessary to guard against the following case: if a feedback packet is not received by all users upon its first transmission, it is retransmitted so that it is possible that a user may receive multiple copies of a “single” feedback packet (meaning that all these packets contain feedback for the same group of slots). Without any additional provisioning, this user cannot distinguish this case from the case of multiple feedback packets that occurred in contiguous groups of slots and happened to experience exactly the same erasures. This problem is solved by enforcing the rule of flipping h_2 between transmission of feedback packets that correspond to *different* groups of slots during the N phases of CODE1_{pub} .

After all log packets have been successfully received, the transmitter broadcasts a final packet with all bits (including h_1 , h_2) set to 0 until it is also received by all users. This packet, which can be easily distinguished by previous feedback log packets since it differs in the h_1 bit, informs the receivers that transmission of all relevant information is complete. The entire overhead scheme is pictorially demonstrated in Fig. 3.

Assuming the order of processing Q_S to be known a priori, each receiver can actually “replay” the execution of CODE1_{pub} , up to the point for which it has received the corresponding part of the feedback log, since it can reproduce the coefficients $a_s(p)$ using the same coefficient generation procedure and linear independence checking procedure (see discussion at the end of the proof of Lemma 5 in the Appendix) as the transmitter. Hence, the receiver can create local copies of the transmitter side queues Q_S and counters K_S^i and use (4) to iteratively compute the $\mathbf{b}_s^{(i)}$, $c_s^{(i)}$ values of each transmitted packet s . The FIFO manner of storing packets at the receiver is crucial, since it associates each received packet to the correct ACK/NACK group. The following result now follows from Theorem 1.

²it is not necessary that any feedback packet is successfully received by all users simultaneously. During the transmission of the feedback log, the transmitter keeps track of which users receive a feedback packet, say by raising a flag whenever a user receives a packet. Hence, the transmitter need transmit a single feedback packet only until the flags for all users have been raised, at which point it starts transmitting the next feedback packet (resetting all flags).

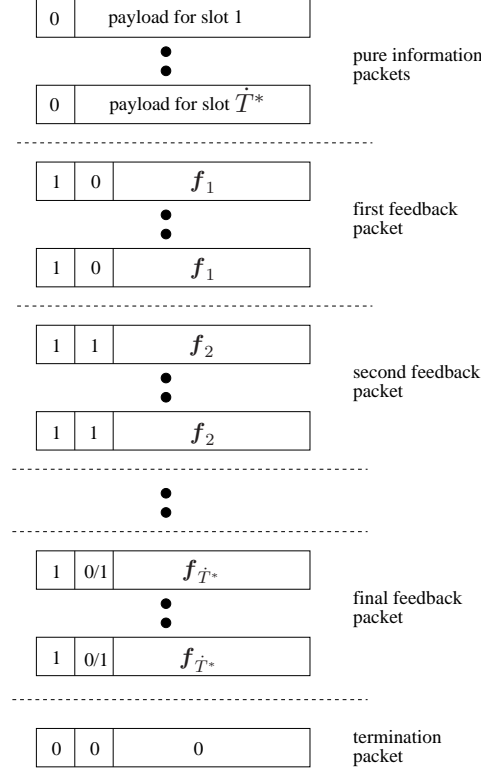


Fig. 3. Distinguishing packets at the receivers based on overhead bits.

Theorem 4: Under the overhead scheme described above, the rate region of CODE1_{pri} , measured in information bits per transmission, for arbitrary channel statistics satisfies the following relation

$$\mathcal{R}_{\text{CODE1}_{pri}} \supseteq \left\{ \mathbf{R} : \sum_{S \subseteq \mathcal{N}} \max_{i \in S} (R_i \hat{f}_S^i) \leq \frac{L-1}{1 + \frac{N^2}{(L-2)(1-\epsilon_{max})}} \right\}, \quad (15)$$

where $\epsilon_{max} = \max_{i \in \mathcal{N}} \epsilon_i$.

$\mathcal{R}_{\text{CODE1}_{pri}}$ approximates $\mathcal{R}_{\text{CODE1}_{pub}}$ within 1 bit as $L \rightarrow \infty$, so that the overhead-induced rate loss is minimal. An example, for $N = 10$ and $\epsilon_{max} = 0.5$ (the latter represents very poor channel conditions; ϵ_{max} is typically much smaller), a length of $L = 8000$ bits leads to a rate loss of 2.5% w.r.t. $\mathcal{R}_{\text{CODE1}_{pub}}$.

Proof: The proof is similar to the proof of Theorem 1 for the case of public feedback, with the important difference that we must now also take into account the number of slots required for the transmission of the feedback log to all users. Based on the description of CODE1_{pri} , the total number of slots \dot{T}^{**} needed by this algorithm is

$$\dot{T}^{**} = \dot{T}^* + \sum_{l=1}^{1 + \lceil N\dot{T}^*/(L-2) \rceil} \dot{N}_l, \quad (16)$$

where the first part in the above sum (i.e. \dot{T}^*) is the number of slots required by CODE1_{pub} and the second part is the total number of slots required to transmit the packetized feedback log (i.e. $1 + \lceil N\dot{T}^*/(L-2) \rceil$ packets, including the termination packet), where $\dot{N}_l \triangleq \max_{i \in \mathcal{N}} \dot{N}_{i,l}$, with $\dot{N}_{i,l}$ the (random) number of transmissions required until the l -th feedback packet is received by user i . It is clear that $\dot{N}_{i,l}$ are geometrically distributed with $\Pr(\dot{N}_{i,l} = \nu) = \epsilon_i^{\nu-1}(1-\epsilon_i)$ while \dot{N}_l are (temporally) iid random variables. The following relations will also be useful.

$$\begin{aligned} \Pr(\dot{N}_l \geq \nu) &= \Pr\left(\bigcup_{i \in \mathcal{N}} \{\dot{N}_{i,l} \geq \nu\}\right) \leq \sum_{i \in \mathcal{N}} \Pr(\dot{N}_{i,l} \geq \nu) = \sum_{i \in \mathcal{N}} \epsilon_i^{\nu-1} \leq N\epsilon_{\max}^{\nu-1}, \\ \mathbb{E}[\dot{N}_l] &= \sum_{\nu=1}^{\infty} \Pr(\dot{N}_l \geq \nu) \leq \sum_{\nu=1}^{\infty} N\epsilon_{\max}^{\nu-1} = \frac{N}{1-\epsilon_{\max}}. \end{aligned} \quad (17)$$

Rewriting (16) as

$$\frac{\dot{T}^{**}}{n} = \frac{\dot{T}^*}{n} + \frac{1 + \lceil N\dot{T}^*/(L-2) \rceil}{n} \left[\frac{1}{1 + \lceil N\dot{T}^*/(L-2) \rceil} \sum_{l=1}^{1 + \lceil N\dot{T}^*/(L-2) \rceil} \dot{N}_l \right], \quad (18)$$

and using (65) of Appendix C for the asymptotic behavior of \dot{T}^*/n as $n \rightarrow \infty$, and the fact that $\dot{T}^* \rightarrow \infty$ w.p. 1 as $n \rightarrow \infty$, so that we can invoke the SLLN for the term inside brackets, we conclude that

$$\bar{T}^{**}(\mathbf{R}) \triangleq \lim_{n \rightarrow \infty} \frac{\dot{T}^{**}}{n} = \left[1 + \frac{N}{L-2} \mathbb{E}[\dot{N}_l] \right] \lim_{n \rightarrow \infty} \frac{\dot{T}^*}{n} \leq \left[1 + \frac{N^2}{(L-2)(1-\epsilon_{\max})} \right] \sum_{\emptyset \neq S \subseteq \mathcal{N}} \max_{i \in S} (\hat{f}_S^i R_i), \quad (19)$$

where we used (17) in the last inequality of the above expression. We can now apply verbatim the argument used in Appendix C (Section C-B) to show that the achievable rate region of CODE1_{pri} , in information symbols per transmission, is

$$\mathcal{R}_{\text{CODE1}_{pri}} = \left\{ \mathbf{R} \geq \mathbf{0} : \bar{T}^{**}(\mathbf{R}) \leq 1 \right\} \supseteq \left\{ \mathbf{R} \geq \mathbf{0} : \left[1 + \frac{N^2}{(L-2)(1-\epsilon_{\max})} \right] \sum_{\emptyset \neq S \subseteq \mathcal{N}} \max_{i \in S} (\hat{f}_S^i R_i) \leq 1 \right\}, \quad (20)$$

where the last set inequality is due to (19). Eq. (15) follows immediately by noting that each transmitted packet in the pure information transmission phase (the feedback packets, although necessary for decoding, only carry feedback information that is independent from the actual message) has an information payload of $L-1$ bits. \blacksquare

VI. ACHIEVING CAPACITY FOR 3 USERS AND ARBITRARY CHANNEL STATISTICS

Although CODE1_{pub} achieves the capacity outer bound of Lemma 4 for some channel statistics (namely, those that satisfy condition $\mathcal{R}_{ord} \supseteq \mathcal{C}^{out}$ in Theorem 2), this is not always true, i.e. for certain channel

statistics there exist rates $\mathbf{R} \in \mathcal{C}^{out}$ that are *not* achievable by CODE1_{pub} . This is easily verified for 3 users as follows: consider the case of equal rates, i.e. $R_i = R$ for all $i \in \{1, 2, 3\}$ (which implies that $|\mathcal{K}_i| = K$ for all i), and assume that it holds

$$\begin{aligned} \epsilon_1 &= \epsilon_2 = \epsilon_3, \\ \epsilon_{\{1,2\}} &> \epsilon_{\{1,3\}} > \epsilon_{\{2,3\}}. \end{aligned} \quad (21)$$

Considering all possible permutations on $\{1, 2, 3\}$ and applying Lemma 4 yields the following bound

$$\mathcal{C}_{eq}^{out} = \left\{ \mathbf{R} : R \left(\frac{1}{1 - \epsilon_1} + \frac{1}{1 - \epsilon_{\{1,2\}}} + \frac{1}{1 - \epsilon_{\{1,2,3\}}} \right) \leq L \right\}. \quad (22)$$

Applying (11) of Theorem 1 to the case of equal rates and using (21) produces, after some algebra,

$$\mathcal{R}_{eq, \text{CODE1}_{pub}} = \left\{ \mathbf{R} : R \left(\frac{1}{1 - \epsilon_1} + \frac{2}{1 - \epsilon_{\{1,2\}}} - \frac{1}{1 - \epsilon_{\{2,3\}}} + \frac{1}{1 - \epsilon_{\{1,2,3\}}} \right) \leq L \right\}, \quad (23)$$

which implies, since $\frac{1}{1 - \epsilon_{\{1,2\}}} > \frac{1}{1 - \epsilon_{\{2,3\}}}$, that $\mathcal{R}_{eq, \text{CODE1}_{pub}} \subset \mathcal{C}_{eq}^{out}$. This demonstrates the suboptimality of CODE1_{pub} .

A more intuitive explanation for the suboptimal performance of CODE1_{pub} under asymmetric channel statistics for the 3-receiver case can also be given through the following argument (note that, for $N = 3$, the network corresponding to Fig. 2 contains only queues for sets $\mathcal{S} \in \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, in addition to $Q_{D_1}, Q_{D_2}, Q_{D_3}$). Assume that in phase 2 of CODE1_{pub} , the order in which the queues are processed is $\{1, 2\}, \{1, 3\}, \{2, 3\}$. When the transmitter sends linear combinations of packets from $Q_{\{1,2\}}$, it is quite possible that the indices $K_{\{1,2\}}^1, K_{\{1,2\}}^2$ do not become zero simultaneously. Say it happens that, at some slot t , it holds $K_{\{1,2\}}^1(t) = 0$ and $K_{\{1,2\}}^2(t) > 0$. By construction, CODE1_{pub} will continue to transmit linear combinations from $Q_{\{1,2\}}$ until $K_{\{1,2\}}^2$ also becomes 0. However, this introduces a degree of inefficiency, as evidenced in step 2 of ACTFB1.

Specifically, if a transmitted packet s is only received by user 1, step 2 will force s to be retransmitted until some user other than 1 receives it, essentially “wasting” this slot. We claim that there exists potential for improvement at this point, by mixing the packets in $Q_{\{1,2\}}$ with the packets in $Q_{\{1,2,3\}}$. Clearly, the first two statements in Lemma 5 are still true, so that each packet stored in either $Q_{\{1,2\}}$ or $Q_{\{1,2,3\}}$ is a token for both users 1,2. Combining this fact with Proposition 1, any linear combination s of the packets in $Q_{\{1,2\}}, Q_{\{1,2,3\}}$ is a token. In fact, since it will be later shown that it is still possible to define sets $\mathcal{B}_{\mathcal{I}}^{(i)}(t), \mathcal{B}_{D_i}(t)$ so that Lemma 6 holds, a proper selection of $a_s(p)$ allows s to become a *Basis* token, in the next slot, for both 1,2 (provided that it holds $K_{\{1,2,3\}}^1 > 0$). Hence, even if the packet is received only by 1, the slot is not wasted, since 1 recovers a *Basis* token.

Unfortunately, the previous reasoning implies that the rule of always combining packets from a single queue must be discarded if the objective is to achieve capacity. For $N > 3$, it is not even clear what structure a capacity achieving algorithm should have. However, for $N = 3$, we present the following algorithm, named CODE2_{pub} , which achieves capacity for arbitrary channels, assuming public feedback is available.

CODE2_{pub} operates in phases as follows. Phase 1 of CODE2_{pub} is identical to phase 1 of CODE1_{pub} , with the transmitter acting according to the rules in ACTFB1 (note that step 2 of ACTFB1 cannot occur in this phase of CODE2_{pub}). In phase 2 of CODE2_{pub} , the transmitter orders the level 2 queues Q_S according to an arbitrary rule and sequentially processes each Q_S by transmitting linear combinations from Q_S until it holds $K_S^i = 0$ for *at least one* user $i \in S$. When this occurs, the transmitter moves to the next level 2 queue. Again, the steps in ACTFB1 are applied. When all level 2 queues have been processed, each such queue Q_S has at most one surviving user index (meaning some $i \in S$ with $K_S^i > 0$). For convenience, we denote this time instant with t_2^* and define the survival number $\dot{S}u(i)$ of index $i \in \{1, 2, 3\}$ as $\dot{S}u(i) \triangleq |\{S : |S| = 2, K_S^i(t_2^*) > 0\}|$. In words, $\dot{S}u(i)$ is equal to the number of level 2 queues which contain unrecovered *Basis* tokens for user i at time t_2^* . Clearly, $\dot{S}u(i)$ is a random variable that depends on the prior erasure events (hence, the dot accent) and satisfies $0 \leq \dot{S}u(i) \leq 2$ for all $i \in \{1, 2, 3\}$. The transmitter now distinguishes cases as follows:

- 1) if it holds $\dot{S}u(i) = 0$ for all $i \in \{1, 2, 3\}$, CODE2_{pub} reverts to CODE1_{pub} , starting at phase 3.
- 2) if it holds $\dot{S}u(i) = 1$ for all $i \in \{1, 2, 3\}$, CODE2_{pub} reverts to CODE1_{pub} and continues processing each Q_S queue in level 2 until all K_S^i become zero.
- 3) otherwise, there exists at least one pair of users i, j such that $\dot{S}u(i) = 0, \dot{S}u(j) > 0$. In this case, simple enumeration reveals that all possible configurations of $\dot{S}u(l)$ for $l \in \{1, 2, 3\}$ fall in exactly one of the following 4 categories:
 - a) there exist distinct users $i^*, j^*, k^* \in \{1, 2, 3\}$ such that $\dot{S}u(i^*) = 0, \dot{S}u(j^*) = 1, \dot{S}u(k^*) = 2$.
 - b) there exist distinct users $i^*, j^*, k^* \in \{1, 2, 3\}$ such that $\dot{S}u(i^*) = 0, \dot{S}u(j^*) = \dot{S}u(k^*) = 1$.
 - c) there exist distinct users $i^*, j^*, k^* \in \{1, 2, 3\}$ such that $\dot{S}u(i^*) = \dot{S}u(j^*) = 0$ and $\dot{S}u(k^*) = 2$.
 - d) there exist distinct users $i^*, j^*, k^* \in \{1, 2, 3\}$ such that $\dot{S}u(i^*) = \dot{S}u(j^*) = 0$ and $\dot{S}u(k^*) = 1$.

To provide some concrete examples, Fig. 4 contains 4 possible configurations (each belonging, from left to right, to one of the above categories), where circles are used to denote surviving indices.

The values (i^*, j^*, k^*) for each configuration are $(3, 2, 1)$, $(2, 1, 3)$, $(3, 2, 1)$, $(3, 2, 1)$, respectively.

We hereafter concentrate on case 3 of the above list, since cases 1, 2 revert to CODE1_{pub} . The transmitter

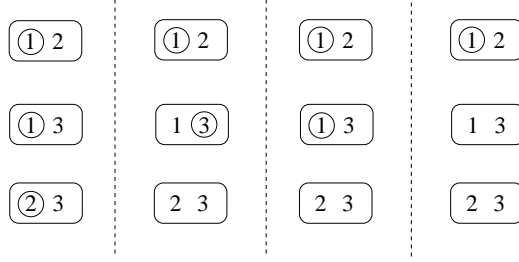


Fig. 4. Possible states of innovative token indices \hat{K}_S^i for the level 2 queues at epoch t_2^* .

now constructs the set $\mathcal{Q}_{\dot{S}_u} = \{Q_{\{i^*, j\}} : \dot{S}_u(i^*) = 0, \hat{K}_{\{i^*, j\}}^j(t_2^*) > 0\}$ consisting of all level 2 queues that contain a surviving index j and an index i^* with $\dot{S}_u(i^*) = 0$. Relative order within $\mathcal{Q}_{\dot{S}_u}$ is unimportant. A subphase, called 2.1, is now initiated, in which the following actions are performed:

- the transmitter processes each queue $Q_{\{i^*, j\}}$ in $\mathcal{Q}_{\dot{S}_u}$ and transmits a packet s which is a linear combination of all packets in queues $Q_{\{i^*, j\}}$ and $Q_{\{1,2,3\}}$ (“and” denotes grouping in this context and should not be interpreted in the Boolean sense). The coefficients $a_s(p)$ are selected such that s is a *Basis* token for j as well as i^* (for the latter case, this is true if it holds $K_{\{1,2,3\}}^{i^*} > 0$). It will be proved in Appendix F that this selection is always possible. Depending on the received feedback, the following actions, collectively referred to as ACTFB2, are taken.

ACTFB2 actions:

- 1) if s is erased by all users, s is retransmitted.
- 2) if s is received only by i^* when it holds $K_{\{1,2,3\}}^{i^*} = 0$, s is retransmitted.
- 3) if j receives s , $K_{\{i^*, j\}}^j$ is decreased by 1 and K_{D_j} is increased by 1.
- 4) if i^* receives s and it holds $K_{\{1,2,3\}}^{i^*} > 0$, $K_{\{1,2,3\}}^{i^*}$ is decreased by 1 and $K_{D_{i^*}}$ is increased by 1.
- 5) if j erases s and $k \in \{1, 2, 3\} - \{i^*, j\}$ receives it, s is added to $Q_{\{1,2,3\}}$, $K_{\{i^*, j\}}^j$ is decreased by 1 and $K_{\{1,2,3\}}^j$ is increased by 1.

Notice that, apart from step 4) in the above list, ACTFB2 is similar to ACTFB1. The above procedure is repeated until it holds $K_{\{i^*, j\}}^j = 0$, at which point the next queue in $\mathcal{Q}_{\dot{S}_u}$ is processed. The above procedure is repeated until all queues in $\mathcal{Q}_{\dot{S}_u}$ have been processed.

- once all queues in $\mathcal{Q}_{\dot{S}_u}$ have been processed, the transmitter computes the new values of $\dot{S}_u(i)$ for $i \in \{1, 2, 3\}$ and constructs $\mathcal{Q}_{\dot{S}_u}$ from scratch. If $\mathcal{Q}_{\dot{S}_u} = \emptyset$, CODE2_{pub} reverts to CODE1_{pub} starting at phase 3, otherwise it repeats the above procedure verbatim for the new $\mathcal{Q}_{\dot{S}_u}$. It can be easily

verified that at most 2 iterations of this procedure will be performed until it holds $\mathcal{Q}_{\dot{S}_u} = \emptyset$.

As a final comment, step 2 of ACTFB2 is similar to step 2 of ACTFB1 so one could argue that CODE2_{pub} still performs inefficiently. However, by construction of $\mathcal{Q}_{\dot{S}_u}$, it is easy to verify that if, during the combination of $Q_{\{i^*,j\}} \in \mathcal{Q}_{\dot{S}_u}$ with $Q_{\{1,2,3\}}$, $K_{\{1,2,3\}}^{i^*}$ becomes 0 before $K_{\{i^*,j\}}^j$ does, then i^* has no more *Basis* tokens to recover (i.e. it holds $K_S^{i^*} = 0$ for all $S \subseteq \mathcal{N}$). Hence, i^* cannot gain any more linearly independent tokens by combining $Q_{\{i^*,j\}}$ with $Q_{\{1,2,3\}}$ and no efficiency is lost.

To provide a concrete example for the last statement, consider the application of subphase 2.1 to the leftmost configuration in Fig. 4. It holds $\mathcal{Q}_{\dot{S}_u} = \{Q_{\{1,3\}}, Q_{\{2,3\}}\}$ and the transmitter starts combining $Q_{\{1,3\}}$ with $Q_{\{1,2,3\}}$ until $K_{\{2,3\}}^2$ becomes 0. If it happens that $K_{\{1,2,3\}}^3$ becomes 0 before $K_{\{2,3\}}^2$, then 3 has indeed recovered all *Basis* tokens so that, even if step 2 occurs, no efficiency gain is possible. The same conclusion is reached by examining the 3 other categories shown in Fig. 4. Hence, at the end of subphase 2.1, it holds $K_S^i = 0$ for all $i \in \mathcal{S}$ with $|\mathcal{S}| = 2$ and CODE2_{pub} reverts to CODE1_{pub} starting at phase 3.

The properties and achievable rate region of CODE2_{pub} can be determined by an approach similar to that of CODE1_{pub} . Specifically, the correctness of CODE2_{pub} is proved in Appendix F, where a slight modification of Lemma 5 is used to show that Lemma 6 is still true for CODE2_{pub} . This guarantees that at the end of CODE2_{pub} , all 3 users have received the required number of linearly independent tokens and can decode their packets. The performance analysis for CODE2_{pub} is identical to CODE1_{pub} , up to time t_2^* . From this point on, the number of tokens produced during the combination of the queues in $\mathcal{Q}_{\dot{S}_u}$ with $Q_{\{1,2,3\}}$ must be carefully computed. The computation is relatively straightforward but lengthy, and is deferred to Appendix G. The final result is:

Theorem 5: CODE2_{pub} achieves the capacity outer bound of \mathcal{C}^{out} , for $L \geq 2$. In case only private feedback is available, we can construct algorithm CODE2_{pri} , based on CODE2_{pub} , using the overhead scheme employed in CODE1_{pri} . The final result is that the rate region of CODE2_{pri} asymptotically differs from the capacity outer bound by 1 bit as $L \rightarrow \infty$.

VII. CONCLUSIONS

This paper presented 2 coding algorithms, CODE1_{pub} and CODE2_{pub} , which achieve (assuming public feedback) an outer bound of the feedback capacity region of the N -user broadcast erasure channel with N unicast sessions for the following cases, respectively: 1) arbitrary N and channel statistics that satisfy the general condition in Theorem 2 (this includes symmetric channels as a special case), and 2) arbitrary channel statistics, for $N = 3$. If public feedback is not available, a simple overhead scheme can be applied

on top of each algorithm, leading to a rate region that asymptotically differs from the outer bound by 1 bit as $L \rightarrow \infty$. The main characteristic of the algorithms is the introduction of virtual queues to store packets, depending on received feedback, and the appropriate mixing of the packets, without requiring any knowledge of channel statistics, to allow for simultaneous reception of innovative packets by multiple users.

Since only an outer bound to the capacity region is known for $N \geq 4$ and arbitrary channels, the search for capacity achieving algorithms for $N \geq 4$ is an obvious future research topic. It is expected that such algorithms cannot be constructed through minor modifications of CODE1_{pub} , as was the case with CODE2_{pub} , and may possibly require complete knowledge of channel statistics. If this is the case, adaptive algorithms that essentially “learn” the relevant statistics may be appropriate. Suboptimal algorithms with guaranteed performance bounds in the spirit of [13] may also be of interest.

APPENDIX A

PROOF OF LEMMA 5

By construction of CODE1_{pub} , the only way a packet s can be stored in queue Q_S , with $|\mathcal{S}| \geq 2$, is during step 4 of ACTFB1 (since, excluding packets that are received by a user $i \in \mathcal{S}$ and moved to queue Q_{D_i} , no packets are moved between queues Q_S in the other steps of ACTFB1). Thus, the execution of step 4 implies that s is a linear combination of packets in some queue $Q_{\mathcal{I}_s}$, with $\emptyset \neq \mathcal{I}_s \subset \mathcal{S}$, and s is received by *all* users in $\mathcal{S} - \mathcal{I}_s$ and erased by *all* users in $\mathcal{N} - \mathcal{S}$. This completes the proof of the first statement.

For the second statement of the Lemma, we note that the algorithm’s operation implies that any transmitted packet s is decomposed as $s = \sum_{u \in \cup_{j \in \mathcal{N}} \mathcal{K}_j} \tilde{a}_s(u)u$ (the algorithm essentially sends linear combinations of linear combinations etc.). Furthermore, we can combine the initialization of CODE1_{pub} (for queues Q_S with $|\mathcal{S}| = 1$) with the first statement in Lemma 5 (proved in the previous paragraph) to show, via strong induction on $|\mathcal{S}| = 2, \dots, N$, that, for all $\mathcal{S} \subseteq \mathcal{N}$ and any packet s stored in Q_S , it holds $\tilde{a}_s(u) = 0$ for all $u \in \mathcal{K}_j$ with $j \notin \mathcal{S}$. Specifically, any $s \in Q_S$ must have entered Q_S during step 4 of ACTFB1, so that it holds $s = \sum_{p \in Q_{\mathcal{I}_s}} a_s(p)p$, where $\mathcal{I}_s \subset \mathcal{S}$. Using the strong induction hypothesis for \mathcal{I}_s , we know that any $p \in Q_{\mathcal{I}_s}$ is written as $p = \sum_{u \in \cup_{j \in \mathcal{I}_s} \mathcal{K}_j} \tilde{a}_p(u)u$. Combining the last two expressions, we conclude that any packet s stored in Q_S can be written as

$$s = \sum_{u \in \cup_{j \in \mathcal{S}} \mathcal{K}_j} \tilde{a}_s(u)u, \quad (24)$$

for suitable $\tilde{a}_s(u)$, and the second statement is also proved.

To prove the third statement of the Lemma, we apply strong induction on $|\mathcal{S}|$, starting with $|\mathcal{S}| = 1$. Due to the initialization of CODE1_{pub} , any packet s stored in $Q_{\{i\}}$ belongs to set \mathcal{K}_i , so that s is a (trivial) token for user i and no other user. We now consider any s stored in queue $Q_{\mathcal{S}}$ with $|\mathcal{S}| > 1$, and use the first statement of the Lemma to write $s = \sum_{p \in Q_{\mathcal{I}_s}} a_s(p)p$, where $\mathcal{I}_s \subset \mathcal{S}$. This also implies that s was received by all users in set $\mathcal{S} - \mathcal{I}_s$, so that s is a token for all users in the set $\mathcal{S} - \mathcal{I}_s$. Combining the inductive hypothesis for set \mathcal{I}_s with Proposition 1, we conclude that s is a token for all $i \in \mathcal{I}_s$ as well, so that s is a token for all $i \in \mathcal{S}$. To show that s is not a token for any $i \notin \mathcal{S}$, we combine the fact that s is a linear combination of packets destined for users in set $\cup_{j \in \mathcal{I}_s} \mathcal{K}_j$ only (second statement of the Lemma) with the fact that s was erased by all users $i \notin \mathcal{S}$ (first statement of Lemma). Hence, i cannot be a token for any $i \notin \mathcal{S}$.

Before we prove the fourth statement in Lemma 5, we need to establish some intermediate results. The following Proposition is easily proved by considering the union bound for the probabilities of the complementary events.

Proposition 2: For any events A_j , with $j = 1, \dots, m$, it holds

$$\Pr(\cap_{j=1}^m A_j) \geq \sum_{j=1}^m \Pr(A_j) - m + 1.$$

The following result will be crucial in proving Lemma 5.

Lemma 8: Let $\{\mathbf{v}_1, \dots, \mathbf{v}_M\}$ be a basis set of the vector space \mathbb{F}_q^M and consider a subspace \mathcal{U} with dimension $l \geq 1$, which contains the set $\{\mathbf{v}_1, \dots, \mathbf{v}_K\}$, with $1 \leq K \leq l$. Then, the subspace $\mathcal{U} \cap \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$ has dimension at most $l - 1$, and $\mathcal{U} - \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$ is a non-empty set. Additionally, for any vector $\mathbf{u} \in \mathcal{U} - \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$, the set $\{\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_M\}$ is a basis of \mathbb{F}_q^M .

Proof: We use contradiction to show that $\dim(\mathcal{U} \cap \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})) \leq l - 1$. Specifically, assume that $\dim(\mathcal{U} \cap \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})) = l$. Then there exists a set $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_l\}$ which forms a basis of $\mathcal{U} \cap \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$. Therefore, $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_l\} \subseteq \mathcal{U} \cap \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$ is a basis of \mathcal{U} as well, since it is a linearly independent set of cardinality l that is contained in the subspace \mathcal{U} of dimension l . The basis property for \mathcal{U} now implies that $\mathbf{v}_1 \in \text{span}(\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_l\})$ and, since $\tilde{\mathbf{v}}_i \in \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$ for $1 \leq i \leq l$, it also holds $\mathbf{v}_1 \in \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$. This contradicts the assumption that $\{\mathbf{v}_1, \dots, \mathbf{v}_M\}$ are linearly independent and proves the desired result. Additionally, since $\mathbf{v}_1 \notin \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$, it also holds $\lambda \mathbf{v}_1 \in \mathcal{U} - \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$ for all $\lambda \in \mathbb{F}_q - \{0\}$, so that $\mathcal{U} - \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\}) \neq \emptyset$.

In order to show that $\{\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_M\}$ is a basis of \mathbb{F}_q^M for any $\mathbf{u} \in \mathcal{U} - \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$, it suffices to show that $\{\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_M\}$ is a linearly independent set. Indeed, pick any $\mathbf{u} \in \mathcal{U} - \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$

and assume that there exist $\alpha, c_i \in \mathbb{F}_q$ such that

$$\alpha \mathbf{u} + \sum_{i=2}^M c_i \mathbf{v}_i = 0.$$

Then it must hold $\alpha = 0$, since the case $\alpha \neq 0$ implies that $\mathbf{u} \in \text{span}(\{\mathbf{v}_2, \dots, \mathbf{v}_M\})$, which is impossible by the selection of \mathbf{u} . The condition $\alpha = 0$ now implies $c_i = 0$, due to the linear independence of $\{\mathbf{v}_2, \dots, \mathbf{v}_M\}$, so that $\{\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_M\}$ is also linearly independent and the proof is complete. \blacksquare

The last intermediate result we need before proving the fourth statement in Lemma 5 is provided below.

Lemma 9: Let \mathbf{v}_j , with $j = 1, \dots, k$, be vectors in \mathbb{F}_q^M . Denote $\mathcal{V} = \text{span}(\{\mathbf{v}_j, j = 1, \dots, k\})$ and $l = \dim(\mathcal{V})$, with $l \geq 1$. Let α_j , with $j = 1, \dots, k$, be independent random variables uniformly distributed in \mathbb{F}_q and construct the random vector $\mathbf{v} = \sum_{j=1}^k \alpha_j \mathbf{v}_j$. Then, \mathbf{v} is uniformly distributed in \mathcal{V} , i.e.

$$\Pr(\mathbf{v} = \mathbf{e}) = \frac{1}{q^l} \quad \forall \mathbf{e} \in \mathcal{V}.$$

Additionally, let $\{\mathbf{b}_1, \dots, \mathbf{b}_M\}$ be a basis of \mathbb{F}_q^M and assume that $\{\mathbf{b}_1, \dots, \mathbf{b}_K\} \subseteq \mathcal{V}$ for $1 \leq K \leq M$. It then holds

$$\Pr(\{\mathbf{v}, \mathbf{b}_2, \dots, \mathbf{b}_M\} \text{ is basis of } \mathbb{F}_q^M) \geq 1 - \frac{1}{q}.$$

Proof: Since \mathcal{V} has dimension l , we can pick l vectors \mathbf{v}_i (out of the k available) as a basis for \mathcal{V} ; without loss of generality, we can permute vector indices so that the basis set is $\{\mathbf{v}_1, \dots, \mathbf{v}_l\}$. Hence, \mathbf{v} can be written as $\mathbf{v} = \sum_{j=1}^l \alpha_j \mathbf{v}_j + \mathbf{g}$, where $\mathbf{g} \triangleq \sum_{j=l+1}^k \alpha_j \mathbf{v}_j$ is a random vector independent from $\sum_{j=1}^l \alpha_j \mathbf{v}_j$. Furthermore, any vector $\mathbf{e} \in \mathcal{V}$ can be written uniquely, through the basis set, as $\mathbf{e} = \sum_{j=1}^l e_j \mathbf{v}_j$. It now holds

$$\begin{aligned} \Pr(\mathbf{v} = \mathbf{e}) &= \sum_{\mathbf{r} \in \mathcal{V}} \Pr\left(\sum_{j=1}^l \alpha_j \mathbf{v}_j + \mathbf{g} = \mathbf{e} \mid \mathbf{g} = \mathbf{r}\right) \Pr(\mathbf{g} = \mathbf{r}) \\ &= \sum_{\mathbf{r} \in \mathcal{V}} \Pr\left(\sum_{j=1}^l \alpha_j \mathbf{v}_j = \mathbf{e} - \mathbf{r} \mid \mathbf{g} = \mathbf{r}\right) \Pr(\mathbf{g} = \mathbf{r}) \\ &= \sum_{\mathbf{r} \in \mathcal{V}} \Pr\left(\sum_{j=1}^l \alpha_j \mathbf{v}_j = \sum_{j=1}^l (e_j - r_j) \mathbf{v}_j\right) \Pr(\mathbf{g} = \mathbf{r}) \\ &= \sum_{\mathbf{r} \in \mathcal{V}} \Pr(\cap_{j=1}^l \{\alpha_j = e_j - r_j\}) \Pr(\mathbf{g} = \mathbf{r}) = \sum_{\mathbf{r} \in \mathcal{V}} \frac{1}{q^l} \Pr(\mathbf{g} = \mathbf{r}) = \frac{1}{q^l}, \end{aligned} \tag{25}$$

where we used the independence of $\sum_{j=1}^l \alpha_j \mathbf{v}_j$ from \mathbf{g} to remove the conditional probability and exploited the facts that $\{\mathbf{v}_1, \dots, \mathbf{v}_l\}$ is a basis set for \mathcal{V} and α_j are independent and uniformly distributed in \mathbb{F}_q .

To prove the second part, we note that

$$\Pr(\{\mathbf{v}, \mathbf{b}_2, \dots, \mathbf{b}_M\} \text{ basis of } \mathbb{F}_q^M) = 1 - \Pr(\mathbf{v} \in \text{span}(\{\mathbf{b}_2, \dots, \mathbf{b}_M\})). \quad (26)$$

For notational convenience, denote $\mathcal{M} = \text{span}(\{\mathbf{b}_2, \dots, \mathbf{b}_M\})$. It now holds

$$\Pr(\mathbf{v} \in \mathcal{M}) = \sum_{\mathbf{r} \in \mathcal{M} \cap \mathcal{V}} \Pr(\mathbf{v} = \mathbf{r}) = \frac{1}{q^L} |\mathcal{M} \cap \mathcal{V}|, \quad (27)$$

where the last equality is due to the uniform distribution of \mathbf{v} in \mathcal{V} . For all vector spaces over a finite field, it also holds $|\mathcal{M} \cap \mathcal{V}| = q^{\dim(\mathcal{M} \cap \mathcal{V})} \leq q^{\dim(\mathcal{V})-1} = q^{L-1}$, where the inequality is due to Lemma 8. Inserting this inequality into (27) produces $\Pr(\mathbf{v} \in \mathcal{M}) \leq 1/q$, whence the desired result follows immediately. ■

We are now in position to prove the fourth statement of Lemma 5. Specifically, recalling the notation of Criterion 2, we assume that there exist sets $\mathcal{B}_{\mathcal{I}}^{(i)}(t)$, $\mathcal{B}_{D_i}(t)$ such that $\mathring{\mathcal{B}} \triangleq \mathcal{B}_{D_i}(t) \cup \bigcup_{\substack{\mathcal{I}: \mathcal{I} \subseteq \mathcal{N} \\ K_{\mathcal{I}}^i(t) > 0}} \mathcal{B}_{\mathcal{I}}^{(i)}(t)$ is a basis of $\mathbb{F}_q^{|\mathcal{K}_i|}$ for all $i \in \mathcal{N}$. Assuming that `CODE1pub` is currently processing Q_S , define the set $\mathcal{R}_S(t) \triangleq \{i \in \mathcal{S} : K_S^i(t) > 0\}$. We need to show that if, for each $i \in \mathcal{R}_S(t)$, we pick an arbitrary vector $\hat{\mathbf{b}}_i \in \mathcal{B}_S^{(i)}(t)$, then there exists a coefficient vector $\mathbf{a}_s = (a_s(p), p \in Q_S)$ such that the vectors $\mathbf{b}_s^{(i)} = \sum_{p \in Q_S} a_s(p) \mathbf{b}_p^{(i)}$, corresponding to the combination $s = \sum_{p \in Q_S} a_s(p) p$, satisfy the following condition

$$\{\mathbf{b}_s^{(i)}\} \cup \mathcal{B}_{D_i}(t) \cup \bigcup_{\substack{\mathcal{I}: \mathcal{I} \subseteq \mathcal{N} \\ K_{\mathcal{I}}^i(t) > 0}} \mathcal{B}_{\mathcal{I}}^{(i)}(t) - \{\hat{\mathbf{b}}_i\} \text{ is basis of } \mathbb{F}_q^{|\mathcal{K}_i|} \quad \forall i \in \mathcal{R}_S(t). \quad (28)$$

The proof is via a standard probabilistic argument. Specifically, consider the case where coefficients \mathbf{a}_s are iid randomly generated according to a uniform distribution in \mathbb{F}_q . For a given user $i \in \mathcal{R}_S(t)$, define the event $A_i \triangleq \{\{\mathbf{b}_s^{(i)}\} \cup \mathring{\mathcal{B}} - \{\hat{\mathbf{b}}_i\} \text{ is basis of } \mathbb{F}_q^{|\mathcal{K}_i|}\}$, whence it follows from Lemma 9 that $\Pr(A_i) \geq 1 - 1/q$. Applying Proposition 2 to the event $\bigcap_{i \in \mathcal{R}_S(t)} A_i$ yields

$$\Pr(\bigcap_{i \in \mathcal{R}_S(t)} A_i) \geq |\mathcal{R}_S(t)| \left(1 - \frac{1}{q}\right) - |\mathcal{R}_S(t)| + 1 \geq 1 - \frac{|\mathcal{R}_S(t)|}{q} \geq 1 - \frac{N}{q}. \quad (29)$$

Selecting $q > N$ (since q can be as large as 2^L , the condition $q > N$ can be satisfied if $L > \log_2 N$) results in a strictly positive probability, which implies that there exist some vectors $\mathbf{b}_s^{(i)}$ that simultaneously satisfy (28) for all $i \in \mathcal{R}_S(t)$. This completes the proof of the fourth statement in Lemma 5.

The previous analysis suggests the following alternative approach to an exhaustive search for generating $a_s(p)$ in accordance with Criterion 2. If the sets $\mathcal{B}_{\mathcal{I}}^{(i)}$, \mathcal{B}_{D_i} are actually stored at the transmitter and receivers, and since Lemma 5 ensures that, for $q > N$, there exist coefficients $a_s(p)$ which satisfy (10) of Criterion 2, then $a_s(p)$ can be generated randomly and uniformly in \mathbb{F}_q (so that (29) holds) followed by an explicit check by the transmitter whether the generated vectors $\mathbf{b}_s^{(i)}$ indeed satisfy (28). If (28) is

violated for at least one $i \in \mathcal{R}_S(t)$, new coefficients are repeatedly created until the condition is satisfied for all $i \in \mathcal{R}_S(t)$. Only then is the packet s actually transmitted, using the most recent coefficients $a_s(p)$. The average number of trials required to find the suitable coefficients is easily computed as $(1 - N/q)^{-1}$.

APPENDIX B

PROOF OF LEMMA 6

Proof is by induction on t . At the beginning of slot $t = 0$, we can satisfy all conditions by choosing for each $i \in \mathcal{N}$ as follows: $\mathcal{B}_{D_i}(0) = \emptyset$, $\mathcal{B}_{\mathcal{I}}^{(i)}(0) = \emptyset$ for $\mathcal{I} \neq \{i\}$ and $\mathcal{B}_{\{i\}}^{(i)}(0) = \text{standard_basis}(\mathbb{F}_q^{|\mathcal{K}_i|})$. We now assume that the inductive hypothesis is true at the beginning of slot t and the queue currently being processed is Q_S . We construct $\mathcal{R}_S(t) = \{i \in \mathcal{S} : K_S^i(t) > 0\}$ and further assume w.l.o.g. that $\mathcal{R}_S(t) \neq \emptyset$ since, in the opposite case, `CODE1pub` will skip processing Q_S and continue to the next queue. Lemma 5 now guarantees that, due to the validity of the hypothesis (i.e. the existence of $\mathcal{B}_{D_i}^{(i)}(t)$, $\mathcal{B}_{\mathcal{I}}^{(i)}(t)$) at the beginning of slot t , we can select vectors $\hat{\mathbf{b}}_i \in \mathcal{B}_{\mathcal{I}}^{(i)}(t)$, for each $i \in \mathcal{R}_S(t)$, and coefficients $a_s(p)$ for the next packet s to be transmitted from Q_S so that (10) of Criterion 2 is satisfied for all $i \in \mathcal{R}_S(t)$.

For each $i \in \mathcal{N} - \mathcal{R}_S(t)$, it either holds $i \notin \mathcal{S}$ or $i \in \mathcal{S}$, $K_S^i(t) = 0$. In both cases, by construction of ACTFB1, the transmission of s does not change any of the $K_{\mathcal{I}}^i$, K_{D_i} indices. Hence, at the beginning of slot $t + 1$, we can select $\mathcal{B}_{\mathcal{I}}^{(i)}(t + 1) = \mathcal{B}_{\mathcal{I}}^{(i)}(t)$, for all $\mathcal{I} \subseteq \mathcal{N}$, and $\mathcal{B}_{D_i}(t + 1) = \mathcal{B}_{D_i}(t)$ so that, for all $i \in \mathcal{N} - \mathcal{R}_S(t)$, the inductive hypothesis holds for $t + 1$ as well. We now concentrate on $i \in \mathcal{R}_S(t)$ and consider the following mutually exclusive cases:

- if i receives s , ACTFB1 forces s to be added to Q_{D_i} and K_S^i to be decreased by one, while K_{D_i} is increased by one. Accordingly, we select $\mathcal{B}_S^{(i)}(t + 1) = \mathcal{B}_S^{(i)}(t) - \{\hat{\mathbf{b}}_i\}$ and $\mathcal{B}_{D_i}(t + 1) = \mathcal{B}_{D_i}(t) \cup \{\mathbf{b}_s^{(i)}\}$, while all other sets $\mathcal{B}_{\mathcal{I}}^{(i)}$ remain unaffected. Lemma 5 now implies that the union of the new sets at slot $t + 1$ form a basis of $\mathbb{F}_q^{|\mathcal{K}_i|}$.
- if i erases s and all users in a maximal set $\mathcal{G} \subseteq \mathcal{N} - \mathcal{S}$ receive s , then K_S^i is decreased by one and $K_{S \cup \mathcal{G}}^i$ is increased by one, according to ACTFB1. We now select $\mathcal{B}_S^{(i)}(t + 1) = \mathcal{B}_S^{(i)}(t) - \{\hat{\mathbf{b}}_i\}$ and $\mathcal{B}_{S \cup \mathcal{G}}^{(i)}(t + 1) = \mathcal{B}_{S \cup \mathcal{G}}^{(i)}(t) \cup \{\mathbf{b}_s^{(i)}\}$ while all other sets remain unchanged. Lemma 5 again implies that the new sets form a basis of $\mathbb{F}_q^{|\mathcal{K}_i|}$ at $t + 1$.
- if i erases s and the only users that receive s belong to a set $\mathcal{G} \subseteq \mathcal{S}$, no $K_{\mathcal{I}}^i$, K_{D_i} indices are affected so that no sets need be changed. In this case, the inductive hypothesis holds trivially at $t + 1$.

Since the above list contains all possible cases, we conclude that the hypothesis is true at the beginning of slot $t + 1$ and the proof is complete.

APPENDIX C
PROOF OF THEOREM 1

A. Some auxiliary results

We first need to establish some additional notation and intermediate results. Denote with $R_{\mathcal{G}} \triangleq \{Z_i = 0, \forall i \in \mathcal{G}\}$ the event that *all* users in set \mathcal{G} receive the transmitted packet, whence it follows from De Morgan's law that

$$R_{\mathcal{G}}^c = \bigsqcup_{\mathcal{H} \neq \emptyset: \mathcal{H} \subseteq \mathcal{G}} (E_{\mathcal{H}} \cap R_{\mathcal{G}-\mathcal{H}}), \quad (30)$$

where c stands for set complement and \bigsqcup denotes a union of disjoint sets. For completeness, we define $E_{\emptyset} = R_{\emptyset} = \Omega$ (the sample space). Introducing the quantity $p_{\mathcal{S}, \mathcal{G}} \triangleq \Pr(E_{\mathcal{S}} \cap R_{\mathcal{G}})$ for all disjoint $\mathcal{S}, \mathcal{G} \subseteq \mathcal{N}$, we can use (30) to convert the expression $\Pr(E_{\mathcal{S}}) = \Pr(E_{\mathcal{S}} \cap R_{\mathcal{G}}) + \Pr(E_{\mathcal{S}} \cap R_{\mathcal{G}}^c)$ into

$$\Pr(E_{\mathcal{S}}) = \Pr(E_{\mathcal{S}} \cap R_{\mathcal{G}}) + \sum_{\mathcal{H} \neq \emptyset: \mathcal{H} \subseteq \mathcal{G}} \Pr(E_{\mathcal{S} \cup \mathcal{H}} \cap R_{\mathcal{G}-\mathcal{H}}) \Leftrightarrow p_{\mathcal{S}, \mathcal{G}} = \epsilon_{\mathcal{S}} - \sum_{\mathcal{H} \neq \emptyset: \mathcal{H} \subseteq \mathcal{G}} p_{\mathcal{S} \cup \mathcal{H}, \mathcal{G}-\mathcal{H}}. \quad (31)$$

Evaluating the last relation for arbitrary \mathcal{S} and $\mathcal{G} = \{j\}$, with $j \notin \mathcal{S}$, yields

$$p_{\mathcal{S}, \{j\}} = \epsilon_{\mathcal{S}} - \epsilon_{\mathcal{S} \cup \{j\}}. \quad (32)$$

The following result provides a general expression for $p_{\mathcal{S}, \mathcal{G}}$.

Lemma 10: For any non-empty disjoint sets $\mathcal{S}, \mathcal{G} \subseteq \mathcal{N}$, it holds

$$p_{\mathcal{S}, \mathcal{G}} = \sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|} \epsilon_{\mathcal{S} \cup \mathcal{H}}. \quad (33)$$

Proof: Proof is by strong induction on $|\mathcal{G}|$. Specifically, for arbitrary \mathcal{S} and $|\mathcal{G}| = 1$ (say, $\mathcal{G} = \{j\}$), (33) becomes

$$p_{\mathcal{S}, \{j\}} = \sum_{\mathcal{H} \subseteq \{j\}} (-1)^{|\mathcal{H}|} \epsilon_{\mathcal{S} \cup \mathcal{H}} = (-1)^0 \epsilon_{\mathcal{S} \cup \emptyset} + (-1)^1 \epsilon_{\mathcal{S} \cup \{j\}}, \quad (34)$$

which is identical to (32). We now assume that (33) is true for *all* \mathcal{S} and *all* \mathcal{G} with $|\mathcal{G}| = 1, \dots, l$ and show that (33) is still true for *all* \mathcal{S} and *all* $\hat{\mathcal{G}}$ with $|\hat{\mathcal{G}}| = l + 1$. Specifically, we can write $\hat{\mathcal{G}} = \{i\} \cup \mathcal{G}$ where $i \notin \mathcal{G}$ and $|\mathcal{G}| = l$, so that we only need to show

$$p_{\mathcal{S}, \hat{\mathcal{G}}} = p_{\mathcal{S}, \mathcal{G} \cup \{i\}} \stackrel{?}{=} \sum_{\mathcal{H} \subseteq \mathcal{G} \cup \{i\}} (-1)^{|\mathcal{H}|} \epsilon_{\mathcal{S} \cup \mathcal{H}}. \quad (35)$$

Since any subset \mathcal{H} of $\mathcal{G} \cup \{i\}$ is either a subset of \mathcal{G} (and therefore does not contain i) or (exclusive or) \mathcal{H} contains i and a, possibly empty, subset $\tilde{\mathcal{H}}$ of \mathcal{G} , the sum in (35) can be written as

$$\begin{aligned} p_{\mathcal{S}, \mathcal{G} \cup \{i\}} &\stackrel{?}{=} \sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|} \epsilon_{\mathcal{S} \cup \mathcal{H}} + \sum_{\tilde{\mathcal{H}} \subseteq \mathcal{G}} (-1)^{|\tilde{\mathcal{H}}|+1} \epsilon_{\mathcal{S} \cup \{i\} \cup \tilde{\mathcal{H}}} \\ &= \sum_{\mathcal{H} \subseteq \mathcal{H}} (-1)^{|\mathcal{H}|} [\epsilon_{\mathcal{S} \cup \mathcal{H}} - \epsilon_{\mathcal{S} \cup \{i\} \cup \mathcal{H}}]. \end{aligned} \quad (36)$$

However, it also holds

$$\begin{aligned} p_{\mathcal{S}, \mathcal{G} \cup \{i\}} &= \Pr(E_{\mathcal{S}} \cap R_{\{i\}} \cap R_{\mathcal{G}}) = \Pr(E_{\mathcal{S}} \cap R_{\mathcal{G}}) - \Pr(E_{\mathcal{S}} \cap E_{\{i\}} \cap R_{\mathcal{G}}) \\ &= \Pr(E_{\mathcal{S}} \cap R_{\mathcal{G}}) - \Pr(E_{\mathcal{S} \cup \{i\}} \cap R_{\mathcal{G}}) = p_{\mathcal{S}, \mathcal{G}} - p_{\mathcal{S} \cup \{i\}, \mathcal{G}}, \end{aligned} \quad (37)$$

Since $|\mathcal{G}| \leq l$, the inductive hypothesis holds for $p_{\mathcal{S}, \mathcal{G}}$, $p_{\mathcal{S} \cup \{i\}, \mathcal{G}}$, whence we conclude that

$$\begin{aligned} p_{\mathcal{S}, \mathcal{G}} &= \sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|} \epsilon_{\mathcal{S} \cup \mathcal{H}}, \\ p_{\mathcal{S} \cup \{i\}, \mathcal{G}} &= \sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|} \epsilon_{\mathcal{S} \cup \{i\} \cup \mathcal{H}}. \end{aligned} \quad (38)$$

Inserting (38) in (37) immediately produces the RHS of (36) and the proof is complete. \blacksquare

An immediate consequence of Lemma 10 is the following result.

Corollary 1: For any $\mathcal{S} \subseteq \mathcal{N}$ with $i \in \mathcal{S}$, the probability that a transmitted packet is received *exactly* by all users in $\mathcal{S} - \{i\}$ (and none other) is given by

$$p_{\mathcal{N} - (\mathcal{S} - \{i\}), \mathcal{S} - \{i\}} = \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i\}} (-1)^{|\mathcal{H}|} \epsilon_{(\mathcal{N} - (\mathcal{S} - \{i\})) \cup \mathcal{H}}.$$

For the next auxiliary result, we need to introduce some further notation. Consider some given n , $\mathbf{R} > \mathbf{0}$ and the application of the original CODE1_{pub} (i.e. without the fixed blocklength modification) for $\mathbf{K} = \lceil n\mathbf{R} \rceil$ packets. We hereafter use consistently a dot accent to explicitly denote a random variable. We denote with $\dot{T}_{i, \mathcal{S}}^*$ the number of slots (viewed as a random variable due to the random erasures) it takes under CODE1_{pub} for index $K_{\mathcal{S}}^i$ to become 0 during the processing of queue $Q_{\mathcal{S}}$, while $\dot{T}_{\mathcal{S}}^*$ (resp. \dot{T}^*) denotes the number of slots it takes under CODE1_{pub} to process queue $Q_{\mathcal{S}}$ (resp. *all* queues). Hence, it holds

$$\begin{aligned} \dot{T}_{\mathcal{S}}^* &= \max_{i \in \mathcal{S}} \dot{T}_{i, \mathcal{S}}^*, \\ \dot{T}^* &= \sum_{\emptyset \neq \mathcal{S} \subseteq \mathcal{N}} \dot{T}_{\mathcal{S}}^*. \end{aligned} \quad (39)$$

Due to the random erasures, the time-varying index $\dot{K}_{\mathcal{S}}^i(t)$ is a random process. We denote with $\tilde{t}_{\mathcal{S}}$ the time when processing of queue $Q_{\mathcal{S}}$ begins and define the random variable $\dot{k}_{\mathcal{S}}^i = \dot{K}_{\mathcal{S}}^i(\tilde{t}_{\mathcal{S}})$ so that, by

the algorithm's initialization, it holds $\dot{k}_{\{i\}}^i = K_i$ w.p. 1. By construction of CODE1_{pub} , when queue Q_S is processed at slot t by transmitting a linear combination s , index $\dot{K}_S^i(t)$ is reduced by one (assuming that $\dot{K}_S^i(t) > 0$) only if s is received by at least one user in set $\mathcal{N} - (\mathcal{S} - \{i\})$ (i.e. received by either i or at least one user in $\mathcal{N} - \mathcal{S}$). Denoting with $\dot{N}_{S,l}^i$ the number of slots in the time interval between the $(l-1)$ -th and the l -th reduction of index \dot{K}_S^i during the processing of Q_S , it clearly follows that³

$$\dot{T}_{i,S}^* = \sum_{l=1}^{\dot{k}_S^i} \dot{N}_{S,l}^i, \quad (40)$$

where $\dot{N}_{S,l}^i$ are iid geometric random variables with $\Pr(\dot{N}_{S,l}^i = \nu) = (a_S^i)^{\nu-1}(1 - a_S^i)$, where $a_S^i = \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}$.

Assuming that packet s is transmitted from Q_S at slot t and $\dot{K}_S^i(t)$ is reduced by 1 at the end of the slot, exactly one of the following two mutually exclusive events occurs: either s is successfully received by i (w.p. $\frac{1-\epsilon_i}{1-a_S^i}$) or s is not received by i but is received by all users in set $\mathcal{T} - \mathcal{S}$ (and erased by all users in $\mathcal{N} - \mathcal{T}$), so that it is placed in queue $Q_{\mathcal{T}}$, with $\mathcal{T} \supset \mathcal{S}$, due to step 4 of ACTFB1 . The latter case occurs with probability $\frac{p_{S \rightarrow \mathcal{T}}^i}{1-a_S^i}$, where

$$p_{S \rightarrow \mathcal{T}}^i = p_{\mathcal{N}-(\mathcal{T}-\{i\}), \mathcal{T}-\mathcal{S}}. \quad (41)$$

Note that the above events occur *provided* that $\dot{K}_S^i(t) > 0$ is actually decreased by 1, so that the corresponding probabilities are actually conditional probabilities. This is the reason for the appearance of the term $(1 - a_S^i)$ in the denominator of both probabilities.

We denote with $\dot{\mathcal{D}}_{S,l}^i \supset \mathcal{S}$ the index set of the queue to which the transmitted packet s is moved after the l -th reduction of index \dot{K}_S^i , with $1 \leq l \leq \dot{k}_S^i$, during the processing of Q_S . Obviously, this is a random variable (hence, the dot) that depends on the exact erasures that occurred during the slot of the l -th reduction. From the previous discussion, it holds $\Pr(\dot{\mathcal{D}}_{S,l}^i = \mathcal{T}) = \frac{p_{S \rightarrow \mathcal{T}}^i}{1-a_S^i}$ for all $\mathcal{T} \supset \mathcal{S}$ and the total number of tokens for user i that were moved into $Q_{\mathcal{T}}$ during the processing of Q_S is

$$\dot{k}_{S \rightarrow \mathcal{T}}^i = \sum_{l=1}^{\dot{k}_S^i} \mathbb{I}[\dot{\mathcal{D}}_{S,l}^i = \mathcal{T}] \quad (42)$$

where

$$\mathbb{I}[\dot{\mathcal{D}}_{S,l}^i = \mathcal{T}] = \begin{cases} 1 & \text{w.p. } \frac{p_{S \rightarrow \mathcal{T}}^i}{1-a_S^i}, \\ 0 & \text{w.p. } 1 - \frac{p_{S \rightarrow \mathcal{T}}^i}{1-a_S^i}. \end{cases} \quad (43)$$

³for consistency, we assume that the 0-th reduction of \dot{K}_S^i occurs at \tilde{t}_S , i.e. when processing of Q_S begins.

Step 4 of ACTFB1 now implies the following recursion for all \mathcal{S} with $|\mathcal{S}| \geq 2$

$$\dot{k}_{\mathcal{S}}^i = \sum_{\substack{\emptyset \neq \mathcal{I} \subset \mathcal{S} \\ i \in \mathcal{S}}} k_{\mathcal{I} \rightarrow \mathcal{S}}^i = \sum_{\substack{\emptyset \neq \mathcal{I} \subset \mathcal{S} \\ i \in \mathcal{S}}} \sum_{l=1}^{k_{\mathcal{I}}^i} \mathbb{I}[\dot{\mathcal{D}}_{\mathcal{I},l}^i = \mathcal{S}], \quad (44)$$

which captures the property that $\dot{k}_{\mathcal{S}}^i$ (i.e. the value of $K_{\mathcal{S}}^i$ at the beginning of processing $Q_{\mathcal{S}}$) is equal to the cumulative number of tokens for user i that were moved to $Q_{\mathcal{S}}$ during the prior processing of queues $Q_{\mathcal{I}}$, for $\mathcal{I} \subset \mathcal{S}$. Rewriting (44) as

$$\frac{\dot{k}_{\mathcal{S}}^i}{n} = \sum_{\substack{\emptyset \neq \mathcal{I} \subset \mathcal{S} \\ i \in \mathcal{S}}} \frac{\dot{k}_{\mathcal{I}}^i}{n} \frac{1}{\dot{k}_{\mathcal{I}}^i} \sum_{l=1}^{k_{\mathcal{I}}^i} \mathbb{I}[\dot{\mathcal{D}}_{\mathcal{I},l}^i = \mathcal{S}], \quad (45)$$

we now state the next result.

Lemma 11: Under the application of `CODE1pub` for $\mathbf{K} = \lceil n\mathbf{R} \rceil$, with $\mathbf{R} > \mathbf{0}$, it holds for all $\mathcal{S} \subseteq \mathcal{N}$ and $i \in \mathcal{S}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\dot{k}_{\mathcal{S}}^i}{n} &= k_{\mathcal{S}}^i \quad a.e. \\ \lim_{n \rightarrow \infty} \frac{\dot{T}_{i,\mathcal{S}}^*}{\dot{k}_{\mathcal{S}}^i} &= \frac{1}{1 - \alpha_{\mathcal{S}}^i} \quad a.e. \end{aligned} \quad (46)$$

where $k_{\mathcal{S}}^i > 0$ are deterministic quantities defined through the recursive relation

$$k_{\mathcal{S}}^i = \sum_{\substack{\emptyset \neq \mathcal{I} \subset \mathcal{S} \\ i \in \mathcal{I}}} \frac{k_{\mathcal{I}}^i}{1 - \epsilon_{\mathcal{N} - (\mathcal{S} - \{i\}), \mathcal{S} - \mathcal{I}}} p_{\mathcal{N} - (\mathcal{S} - \{i\}), \mathcal{S} - \mathcal{I}} \quad \forall \mathcal{S} : |\mathcal{S}| \geq 2, \quad (47)$$

and the initial condition $k_{\{i\}}^i = R_i$.

Proof: Proof is by strong induction on $|\mathcal{S}|$. For $|\mathcal{S}| = 1$, the initialization of the algorithm implies that $\dot{k}_{\{i\}}^i = K_i = \lceil nR_i \rceil$, whence we conclude that $\dot{k}_{\{i\}}^i/n \rightarrow R_i$ a.e. as $n \rightarrow \infty$. Additionally, it holds $\dot{T}_{i,\{i\}}^* = \sum_{l=1}^{\dot{k}_{\{i\}}^i} \dot{N}_{\{i\},l}^i$ so that the SLLN yields

$$\frac{1}{\dot{k}_{\{i\}}^i} \dot{T}_{i,\{i\}}^* = \frac{1}{\dot{k}_{\{i\}}^i} \sum_{l=1}^{\dot{k}_{\{i\}}^i} \dot{N}_{\{i\},l}^i \rightarrow \mathbb{E}[\dot{N}_{\{i\}}^i] = \frac{1}{1 - a_{\{i\}}^i} \quad a.e. \text{ as } n \rightarrow \infty, \quad (48)$$

since $\dot{k}_{\{i\}}^i \rightarrow \infty$ a.e. as $n \rightarrow \infty$.

We now assume that (46) is true for all \mathcal{S} with $|\mathcal{S}| \leq m$. Applying (45) to any \mathcal{S} with $|\mathcal{S}| = m + 1$, taking a limit as $n \rightarrow \infty$ and using the inductive hypothesis for all $\mathcal{I} \subset \mathcal{S}$ (since it holds $|\mathcal{I}| \leq m$) and the SLLN (since $\dot{k}_{\mathcal{I}}^i \rightarrow \infty$ a.e. as $n \rightarrow \infty$ and the indicator functions are iid random variables), we arrive at

$$\lim_{n \rightarrow \infty} \frac{\dot{k}_{\mathcal{S}}^i}{n} = \sum_{\substack{\emptyset \neq \mathcal{I} \subset \mathcal{S} \\ i \in \mathcal{S}}} \left(\lim_{n \rightarrow \infty} \frac{\dot{k}_{\mathcal{I}}^i}{n} \right) \mathbb{E}[\mathbb{I}[\dot{\mathcal{D}}_{\mathcal{I},l}^i = \mathcal{S}]] = \sum_{\substack{\emptyset \neq \mathcal{I} \subset \mathcal{S} \\ i \in \mathcal{S}}} k_{\mathcal{I}}^i \frac{p_{\mathcal{I} \rightarrow \mathcal{S}}^i}{1 - a_{\mathcal{I}}^i} = k_{\mathcal{S}}^i. \quad (49)$$

Using (41) to substitute for $p_{\mathcal{I} \rightarrow \mathcal{S}}^i$, $a_{\mathcal{S}}^i$, (49) reduces to (47) for all \mathcal{S} with $|\mathcal{S}| = m + 1$, so that the induction is complete for the first equation in (46). To prove the second equation in (46) for all \mathcal{S} with $|\mathcal{S}| = m + 1$, we follow a procedure similar to the case of $|\mathcal{S}| = 1$ so that

$$\dot{T}_{i,\mathcal{S}}^* = \sum_{l=1}^{k_{\mathcal{S}}^i} \dot{N}_{\mathcal{S},l}^i \Rightarrow \frac{\dot{T}_{i,\mathcal{S}}^*}{k_{\mathcal{S}}^i} = \frac{1}{k_{\mathcal{S}}^i} \sum_{l=1}^{k_{\mathcal{S}}^i} \dot{N}_{\mathcal{S},l}^i \rightarrow \mathbb{E}[\dot{N}_{\mathcal{S}}^i] = \frac{1}{1 - a_{\mathcal{S}}^i} \quad a.e. \text{ as } n \rightarrow \infty, \quad (50)$$

This proves the second equation in (46) and completes the proof. \blacksquare

Using Lemma 11 and rewriting (39) as

$$\begin{aligned} \frac{\dot{T}_{\mathcal{S}}^*}{n} &= \max_{i \in \mathcal{S}} \left(\frac{\dot{T}_{i,\mathcal{S}}^*}{n} \right) = \max_{i \in \mathcal{S}} \left(\frac{\dot{T}_{i,\mathcal{S}}^*}{k_{\mathcal{S}}^i} \frac{k_{\mathcal{S}}^i}{n} \right), \\ \frac{\dot{T}_{\mathcal{S}}^*}{n} &= \sum_{\emptyset \neq \mathcal{S} \subseteq \mathcal{N}} \frac{\dot{T}_{\mathcal{S}}^*}{n}, \end{aligned} \quad (51)$$

we can take a limit as $n \rightarrow \infty$, use (46) and exploit the continuity of \max to pass the limit through it and arrive at the following Corollary.

Corollary 2: Under the application of CODE1_{pub} , it holds

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\dot{T}_{\mathcal{S}}^*}{n} &= \max_{i \in \mathcal{S}} \left(\frac{k_{\mathcal{S}}^i}{1 - a_{\mathcal{S}}^i} \right) \quad a.e. \\ \lim_{n \rightarrow \infty} \frac{\dot{T}_{\mathcal{S}}^*}{n} &= \sum_{\emptyset \neq \mathcal{S} \subseteq \mathcal{N}} \max_{i \in \mathcal{S}} \left(\frac{k_{\mathcal{S}}^i}{1 - a_{\mathcal{S}}^i} \right) \quad a.e. \end{aligned} \quad (52)$$

The last auxiliary result is an explicit solution of (47) (along with the initial condition $k_{\{i\}}^i = R_i$) which, introducing the variable

$$f_{\mathcal{S}}^i \triangleq \frac{k_{\mathcal{S}}^i}{R_i(1 - \epsilon_{\mathcal{N} - (\mathcal{S} - \{i\})})}, \quad (53)$$

is cast into the more convenient form

$$f_{\mathcal{S}}^i = \frac{1}{1 - \epsilon_{\mathcal{N} - (\mathcal{S} - \{i\})}} \sum_{\substack{\emptyset \neq \mathcal{I} \subseteq \mathcal{S} \\ i \in \mathcal{I}}} f_{\mathcal{I}}^i p_{\mathcal{N} - (\mathcal{S} - \{i\}), \mathcal{S} - \mathcal{I}} \quad \forall \mathcal{S} : |\mathcal{S}| \geq 2, \quad (54)$$

with an initial condition of $f_{\{i\}}^i = \frac{1}{1 - \epsilon_{\mathcal{N}}}$. The following Lemma provides an explicit representation of $f_{\mathcal{S}}^i$ and shows that $f_{\mathcal{S}}^i$ is identical to the quantity $\hat{f}_{\mathcal{S}}^i$ introduced in Theorem 1.

Lemma 12: For any set $\mathcal{S} \subseteq \mathcal{N}$ with $i \in \mathcal{S}$, it holds

$$f_{\mathcal{S}}^i = \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i\}} \frac{(-1)^{|\mathcal{H}|}}{1 - \epsilon_{(\mathcal{N} - (\mathcal{S} - \{i\})) \cup \mathcal{H}}}. \quad (55)$$

Proof: The following equivalent expression can be derived from Lemma 10.

$$\begin{aligned} p_{\mathcal{S}, \mathcal{G}} &= \sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|} \epsilon_{\mathcal{S} \cup \mathcal{H}} = \sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|} (1 - (1 - \epsilon_{\mathcal{S} \cup \mathcal{H}})) \\ &= \sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|} + \sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|+1} (1 - \epsilon_{\mathcal{S} \cup \mathcal{H}}) = \sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|+1} (1 - \epsilon_{\mathcal{S} \cup \mathcal{H}}), \end{aligned} \quad (56)$$

where we used the binomial theorem to compute $\sum_{\mathcal{H} \subseteq \mathcal{G}} (-1)^{|\mathcal{H}|} = \sum_{r=0}^{|\mathcal{G}|} \binom{|\mathcal{G}|}{r} (-1)^r = (1-1)^{|\mathcal{G}|} = 0$.

We initially manipulate (54) by substituting for $p_{\mathcal{N}-(\mathcal{S}-\{i\}), \mathcal{S}-\mathcal{I}}$ through (56), which yields

$$f_{\mathcal{S}}^i = \frac{1}{1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}} \sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}} f_{\mathcal{I}}^i \sum_{\mathcal{H} \subseteq \mathcal{S}-\mathcal{I}} (-1)^{|\mathcal{H}|+1} (1 - \epsilon_{(\mathcal{N}-(\mathcal{S}-\{i\})) \cup \mathcal{H}}). \quad (57)$$

Extracting the $\mathcal{H} = \emptyset$ term from the summation over \mathcal{H} yields

$$\begin{aligned} f_{\mathcal{S}}^i &= \frac{1}{1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}} \sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}} f_{\mathcal{I}}^i \left[-(1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}) + \sum_{\emptyset \neq \mathcal{H} \subseteq \mathcal{S}-\mathcal{I}} (-1)^{|\mathcal{H}|+1} (1 - \epsilon_{(\mathcal{N}-(\mathcal{S}-\{i\})) \cup \mathcal{H}}) \right] \\ &= - \sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}} f_{\mathcal{I}}^i + \frac{1}{1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}} \sum_{\emptyset \neq \mathcal{H} \subseteq \mathcal{S}-\{i\}} \sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}-\mathcal{H}} f_{\mathcal{I}}^i (-1)^{|\mathcal{H}|+1} (1 - \epsilon_{(\mathcal{N}-(\mathcal{S}-\{i\})) \cup \mathcal{H}}), \end{aligned} \quad (58)$$

where we changed the order of summation in the second sum of the last line. Moving the first sum in the RHS of the last expression to the LHS produces

$$\sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}} f_{\mathcal{I}}^i = \frac{1}{1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}} \sum_{\emptyset \neq \mathcal{H} \subseteq \mathcal{S}-\{i\}} (-1)^{|\mathcal{H}|+1} (1 - \epsilon_{(\mathcal{N}-(\mathcal{S}-\{i\})) \cup \mathcal{H}}) \left[\sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}-\mathcal{H}} f_{\mathcal{I}}^i \right], \quad (59)$$

which provides a new recursion w.r.t. the term $\sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}} f_{\mathcal{I}}^i$.

For a fixed i , we can use induction on $|\mathcal{S}|$ to show the following relation

$$\sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}} f_{\mathcal{I}}^i = \frac{1}{1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}}, \quad \forall \mathcal{S}, \forall i \in \mathcal{S}. \quad (60)$$

Indeed, for $|\mathcal{S}| = 1$, which implies $\mathcal{S} = \{i\}$, (60) yields $f_{\{i\}}^i = \frac{1}{1 - \epsilon_{\mathcal{N}}}$, which is identical to the initial condition of (54). We now assume that (60) is true for all \mathcal{S} with $|\mathcal{S}| \leq l$ and show that it is also true for all \mathcal{S} with $|\mathcal{S}| = l + 1$. Specifically, for any \mathcal{S} with $|\mathcal{S}| = l + 1$, (59) becomes

$$\begin{aligned} \sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}} f_{\mathcal{I}}^i &= \frac{1}{1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}} \sum_{\emptyset \neq \mathcal{H} \subseteq \mathcal{S}-\{i\}} (-1)^{|\mathcal{H}|+1} (1 - \epsilon_{(\mathcal{N}-(\mathcal{S}-\{i\})) \cup \mathcal{H}}) \frac{1}{1 - \epsilon_{(\mathcal{N}-(\mathcal{S}-\{i\})) \cup \mathcal{H}}} \\ &= \frac{1}{1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}}, \end{aligned} \quad (61)$$

where we used the inductive hypothesis for the terms $\sum_{\{i\} \subseteq \mathcal{I} \subseteq \mathcal{S}-\mathcal{H}} f_{\mathcal{I}}^i$, since $|\mathcal{I}| \leq |\mathcal{S} - \mathcal{H}| \leq l$ when $\mathcal{H} \neq \emptyset$, and applied the binomial theorem. This completes the induction and proves (60).

We denote with $\hat{p}_{\mathcal{I}} \triangleq p_{\mathcal{N}-\mathcal{I},\mathcal{I}}$ the probability that a packet is received by exactly the users in \mathcal{I} (and none other), whence we deduce the following relation

$$\sum_{\mathcal{I} \subseteq \mathcal{S}} \hat{p}_{\mathcal{I}} = \Pr \left(\bigcup_{\mathcal{I} \subseteq \mathcal{S}} (E_{\mathcal{N}-\mathcal{I}} \cap R_{\mathcal{I}}) \right) = \Pr(E_{\mathcal{N}-\mathcal{S}}) = \epsilon_{\mathcal{N}-\mathcal{S}}, \quad (62)$$

which is true for *any* $\mathcal{S} \subseteq \mathcal{N}$. Hence, it also holds $\sum_{\mathcal{I} \subseteq \mathcal{S}-\{i\}} \hat{p}_{\mathcal{I}} = \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}$, so that the following is true for all \mathcal{S} and $i \in \mathcal{S}$

$$\begin{aligned} \sum_{\mathcal{I} \subseteq \mathcal{S}-\{i\}} \hat{p}_{\mathcal{I}} &= \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}, \\ \sum_{\mathcal{I} \subseteq \mathcal{S}-\{i\}} f_{\mathcal{I} \cup \{i\}}^i &= \frac{1}{1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}}, \end{aligned} \quad (63)$$

along with the initial conditions $\hat{p}_{\emptyset} = \epsilon_{\mathcal{N}}$, $f_{\{i\}}^i = 1/(1 - \epsilon_{\mathcal{N}})$. The second equation in (63) is essentially a rewrite of (60).

We now make the crucial observation that (63) allows for a separate recursive computation of $f_{\mathcal{S}}^i$, $\hat{p}_{\mathcal{S}}$ based on the corresponding initial condition. Since the only difference between the two recursions is the RHS term (the recursion for $\hat{p}_{\mathcal{I}}$, $f_{\mathcal{I} \cup \{i\}}^i$ uses $\epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})}$, $(1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})})^{-1}$, respectively), we conclude that any relation that holds for $\hat{p}_{\mathcal{I}}$ also holds for $f_{\mathcal{I} \cup \{i\}}^i$ via a substitution $\epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})} \rightarrow (1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})})^{-1}$. Combining the last statement with Corollary 1 (which provides an expression for $\hat{p}_{\mathcal{S}-\{i\}}$), yields

$$f_{\mathcal{S}}^i = f_{(\mathcal{S}-\{i\}) \cup \{i\}}^i = \sum_{\mathcal{H} \subseteq \mathcal{S}-\{i\}} \frac{(-1)^{|\mathcal{H}|}}{1 - \epsilon_{(\mathcal{N}-(\mathcal{S}-\{i\})) \cup \mathcal{H}}}, \quad (64)$$

which completes the proof. ■

B. Proof of Theorem 1

We are now in position to finally prove Theorem 1. Through a change of variable $\mathcal{H}' = (\mathcal{S} - \{i\}) - \mathcal{H}$, (55) can also be written as $f_{\mathcal{S}}^i = \sum_{\mathcal{H}' \subseteq \mathcal{S}-\{i\}} \frac{(-1)^{|\mathcal{S}|-|\mathcal{H}'|-1}}{1 - \epsilon_{\mathcal{N}-\mathcal{H}'}} = \hat{f}_{\mathcal{S}}^i$. Additionally, using Lemma 12 and (53) to substitute for $k_{\mathcal{S}}^i$ in (52) yields

$$\begin{aligned} \bar{T}_{\mathcal{S}}^*(\mathbf{R}) &\triangleq \lim_{n \rightarrow \infty} \frac{\dot{T}_{\mathcal{S}}^*}{n} = \max_{i \in \mathcal{S}} \left(\frac{\hat{f}_{\mathcal{S}}^i (1 - \epsilon_{\mathcal{N}-(\mathcal{S}-\{i\})})}{1 - a_{\mathcal{S}}^i} \right) = \max_{i \in \mathcal{S}} (\hat{f}_{\mathcal{S}}^i R_i), \\ \bar{T}^*(\mathbf{R}) &\triangleq \lim_{n \rightarrow \infty} \frac{\dot{T}^*}{n} = \sum_{\emptyset \neq \mathcal{S} \subseteq \mathcal{N}} \bar{T}_{\mathcal{S}}^*(\mathbf{R}) = \sum_{\emptyset \neq \mathcal{S} \subseteq \mathcal{N}} \max_{i \in \mathcal{S}} (\hat{f}_{\mathcal{S}}^i R_i), \end{aligned} \quad (65)$$

where we also used (41) to substitute for $a_{\mathcal{S}}^i$. We now show that the achievable region of CODE1_{pub}, in information symbols per transmission, is given by

$$\mathcal{R}_{\text{CODE1}_{\text{pub}}} = \{ \mathbf{R} : \bar{T}^*(\mathbf{R}) \leq 1 \}. \quad (66)$$

The reader can verify that (66) readily yields (11) through (65), considering the fact that each symbol/packet contains L bits. Hence, it remains to prove (66), which is equivalent to proving the following statements: 1) any \mathbf{R} such that $\bar{T}^*(\mathbf{R}) < 1$ is achievable by CODE1_{pub} , and 2) no \mathbf{R} with $\bar{T}^*(\mathbf{R}) > 1$ is achievable by CODE1_{pub} .

To prove the first part of (66), consider any $\mathbf{R} \neq \mathbf{0}$ with $\bar{T}^*(\mathbf{R}) < 1$ and apply the fixed blocklength version of CODE1_{pub} (i.e. stop after n transmissions), with $\mathbf{K} = \lceil n\mathbf{R} \rceil$. By construction of the modified CODE1_{pub} , an error occurs iff $\dot{T}^* > n$. Hence, the probability of error for the modified CODE1_{pub} is

$$p_n(e) = \Pr(\dot{T}^* > n) = \Pr\left(\frac{\dot{T}^*}{n} > 1\right) = \Pr\left(\frac{\dot{T}^*}{n} - \bar{T}^*(\mathbf{R}) > 1 - \bar{T}^*(\mathbf{R})\right). \quad (67)$$

Letting $n \rightarrow \infty$, the relation $\bar{T}^*(\mathbf{R}) < 1$ implies, through (67), that $p_n(e) \rightarrow 0$, since the LHS of the inequality in the last event in (67) goes to 0 as $n \rightarrow \infty$, while the RHS is strictly positive. This proves the first part of (66). A similar argument can be used to show that $\bar{T}^*(\mathbf{r}) > 1$ implies $\lim_{n \rightarrow \infty} p_n(e) = 1$, which proves the second part of (66).

APPENDIX D

PROOF OF LEMMA 7 AND THEOREM 2

Consider an arbitrary $\mathbf{R} \in \mathcal{R}_{ord}$ and define the set

$$\Phi_{\tilde{\pi}}(j) \triangleq \{k \in \mathcal{N} : \tilde{\pi}(k) \geq j\}, \quad (68)$$

where $\tilde{\pi}$ is the permutation corresponding to \mathbf{R} via (12). Additionally, there exists the functional inverse $\tilde{\pi}^{-1}$ of $\tilde{\pi}$ (since $\tilde{\pi}$ is a bijection on \mathcal{N}), which is a permutation on \mathcal{N} as well. In fact, the introduction of $\tilde{\pi}^{-1}$ allows us to rewrite (68) as

$$\Phi_{\tilde{\pi}}(j) = \{\tilde{\pi}^{-1}(j), \tilde{\pi}^{-1}(j+1), \dots, \tilde{\pi}^{-1}(N)\}, \quad (69)$$

which can be proved by standard bidirectional set inclusion. It now holds

$$\sum_{S \subseteq \mathcal{N}} \max_{i \in S} (\hat{f}_S^i R_i) = \sum_{l=1}^N \sum_{S: l = \arg \max_{i \in S} (\hat{f}_S^i R_i)} \hat{f}_S^l R_l = \sum_{j=1}^N R_{\tilde{\pi}^{-1}(j)} \sum_{S: \tilde{\pi}^{-1}(j) = \arg \max_{i \in S} (\hat{f}_S^i R_i)} \hat{f}_S^{\tilde{\pi}^{-1}(j)}, \quad (70)$$

where the last equality follows from the substitution $l = \tilde{\pi}^{-1}(j)$. Since $\mathbf{R} \in \mathcal{R}_{ord}$, (12) now implies

$$\left\{ S : \arg \max_{i \in S} (\hat{f}_S^i R_i) = \tilde{\pi}^{-1}(j) \right\} = \left\{ S : \tilde{\pi}^{-1}(j) = \arg \min_{i \in S} (\tilde{\pi}(i)) \right\}, \quad (71)$$

so that the inner sum in the RHS of (70) becomes

$$\sum_{S: \tilde{\pi}^{-1}(j) = \arg \min_{i \in S} (\tilde{\pi}(i))} \hat{f}_S^{\tilde{\pi}^{-1}(j)} = \sum_{S: \{\tilde{\pi}^{-1}(j)\} \subseteq S \subseteq \Phi_{\tilde{\pi}}(j)} \hat{f}_S^{\tilde{\pi}^{-1}(j)} = \frac{1}{1 - \epsilon_{\mathcal{N} - (\Phi_{\tilde{\pi}}(j) - \{\tilde{\pi}^{-1}(j)\})}}, \quad (72)$$

where the first equality follows from the fact that, by construction, all sets \mathcal{S} appearing in the summation of (72) satisfy the relation

$$\{\tilde{\pi}^{-1}(j)\} \subseteq \mathcal{S} \subseteq \{k \in \mathcal{N} : \tilde{\pi}(k) \geq \tilde{\pi}(\tilde{\pi}^{-1}(j))\} = \Phi_{\tilde{\pi}}(j), \quad (73)$$

and the second equality follows from (60).

The definition of $\Phi_{\tilde{\pi}}(j)$ now implies

$$\mathcal{N} - (\Phi_{\tilde{\pi}}(j) - \{\tilde{\pi}^{-1}(j)\}) = \{k \in \mathcal{N} : \tilde{\pi}(k) \leq j\} = \{\tilde{\pi}^{-1}(1), \dots, \tilde{\pi}^{-1}(j)\}, \quad (74)$$

which can again be proved by bidirectional set inclusion. Inserting (74) into (72) and (70) finally yields

$$\sum_{\mathcal{S} \subseteq \mathcal{N}} \max_{i \in \mathcal{S}} (\hat{f}_{\mathcal{S}}^i R_i) = \sum_{j=1}^N \frac{R_{\tilde{\pi}^{-1}(j)}}{1 - \epsilon_{\{\tilde{\pi}^{-1}(1), \dots, \tilde{\pi}^{-1}(j)\}}}, \quad (75)$$

which completes the proof of Lemma 7.

Regarding Theorem 2, we can prove that $\mathcal{R}_{\text{CODE1}_{pub}} \cap \mathcal{R}_{ord} = \mathcal{C}^{out} \cap \mathcal{R}_{ord}$ by showing that $\mathcal{R}_{\text{CODE1}_{pub}} \cap \mathcal{R}_{ord} \supseteq \mathcal{C}^{out} \cap \mathcal{R}_{ord}$ (the inclusion in the other direction follows trivially from the fact $\mathcal{R}_{\text{CODE1}_{pub}} \subseteq \mathcal{C}^{out}$). Indeed, pick any $\mathbf{R} \in \mathcal{C}^{out} \cap \mathcal{R}_{ord}$. Since $\mathbf{R} \in \mathcal{C}^{out}$, Lemma 4 implies that it holds

$$\max_{\pi \in \mathcal{P}} \left(\sum_{i=1}^N \frac{R_{\pi(i)}}{1 - \epsilon_{\{\pi(1), \dots, \pi(i)\}}} \leq L \right), \quad (76)$$

where \mathcal{P} is the set of all possible permutations on \mathcal{N} , so that \mathcal{P} includes both $\tilde{\pi}$ and $\tilde{\pi}^{-1}$. Hence, (76) also holds for the specific permutation $\tilde{\pi}$ (corresponding to the chosen \mathbf{R}), which implies through (75) and Theorem 1 that $\mathbf{R} \in \mathcal{R}_{\text{CODE1}_{pub}}$. Since \mathbf{R} also belongs to \mathcal{R}_{ord} , it follows that $\mathcal{R}_{\text{CODE1}_{pub}} \cap \mathcal{R}_{ord} \supseteq \mathcal{C}^{out} \cap \mathcal{R}_{ord}$. This completes the proof of the first statement in Theorem 2.

The second statement of Theorem 2 now follows from the fact that the assumption $\mathcal{R}_{ord} \supseteq \mathcal{C}^{out}$ (which also implies $\mathcal{R}_{ord} \supseteq \mathcal{R}_{\text{CODE1}_{pub}}$) transforms the established relation $\mathcal{R}_{\text{CODE1}_{pub}} \cap \mathcal{R}_{ord} = \mathcal{C}^{out} \cap \mathcal{R}_{ord} = \mathcal{D}$ into $\mathcal{R}_{\text{CODE1}_{pub}} = \mathcal{C}^{out} = \mathcal{D}$. Hence, CODE1_{pub} achieves capacity in this case.

APPENDIX E

PROOF OF THEOREM 3

For symmetric channels, we introduce the notation $\tilde{\epsilon}_{|\mathcal{I}|} = \epsilon_{\mathcal{I}}$ for all $\mathcal{I} \subseteq \mathcal{N}$ with a given $|\mathcal{I}|$. It then holds $\tilde{\epsilon}_1 \geq \dots \geq \tilde{\epsilon}_N$, which in turn implies $\frac{1}{1-\tilde{\epsilon}_1} \geq \dots \geq \frac{1}{1-\tilde{\epsilon}_N}$. A simple index exchange argument in Lemma 4 reveals that \mathcal{C}^{out} can be written as

$$\mathcal{C}^{out} = \left\{ \mathbf{R} \geq \mathbf{0} : \sum_{i=1}^N \frac{R_{\hat{\pi}^{-1}(i)}}{1 - \tilde{\epsilon}_i} \leq L \right\}, \quad (77)$$

where $\hat{\pi}$ is the permutation on \mathcal{N} that rearranges \mathbf{R} in non-decreasing order, i.e. $R_{\hat{\pi}^{-1}(1)} \geq \dots \geq R_{\hat{\pi}^{-1}(N)}$.

By definition of symmetric channels, it also holds

$$f_S^i = \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i\}} \frac{(-1)^{|\mathcal{S}| - |\mathcal{H}| - 1}}{1 - \epsilon_{\mathcal{N} - \mathcal{H}}} = \sum_{m=0}^{|\mathcal{S}|-1} \binom{|\mathcal{S}| - 1}{m} \frac{(-1)^{|\mathcal{S}| - m - 1}}{1 - \tilde{\epsilon}_{N-m}},$$

where we used the fact that there exist $\binom{|\mathcal{S}| - 1}{m}$ subsets \mathcal{H} of $\mathcal{S} - \{i\}$ with cardinality m . Hence, f_S^i is independent of i , so that for all $\mathbf{R} \geq \mathbf{0}$ it holds

$$\arg \max_{i \in \mathcal{S}} (f_S^i R_i) = \arg \max_{i \in \mathcal{S}} (R_i) = \arg \min_{i \in \mathcal{S}} (\tilde{\pi}(i)), \quad (78)$$

where the last equality is due to the definition of $\tilde{\pi}$. Hence, it holds $\mathcal{R}_{ord} = \{\mathbf{R} : \mathbf{R} \geq \mathbf{0}\}$ since we can select, for each $\mathbf{R} \geq \mathbf{0}$, the permutation $\tilde{\pi} = \hat{\pi}$ to satisfy (12). Since $\mathcal{R}_{ord} \supseteq \mathcal{C}^{out}$, CODE1_{pub} achieves capacity for symmetric channels and its rate region is given by (77).

In the case of one-sided fair spatially independent channels, we must show that any vector $\mathbf{R} \in \mathcal{R}_{fair}$, i.e. any vector which satisfies

$$\begin{aligned} \epsilon_1 &\geq \dots \geq \epsilon_N, \\ \epsilon_1 R_1 &\geq \dots \geq \epsilon_N R_N, \end{aligned} \quad (79)$$

also belongs to \mathcal{R}_{ord} , i.e. there exists a permutation $\tilde{\pi}$ such that it holds $\arg \max_{i \in \mathcal{S}} (\hat{f}_S^i R_i) = \arg \min_{i \in \mathcal{S}} (\tilde{\pi}(i))$ for all $\mathcal{S} \subseteq \mathcal{N}$. In fact, we will show that the required permutation $\tilde{\pi}$ is the identity permutation; in other words, we will prove that (79) implies $f_S^i R_i \geq f_S^j R_j$ for all $i, j \in \mathcal{S}$ with $i < j$.

Consider an arbitrary set $\mathcal{S} \subseteq \mathcal{N}$ and let $i, j \in \mathcal{S}$. Using Lemma 12 and exploiting the spatial independence, we compute f_S^i as

$$\begin{aligned} f_S^i &= \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i\}} \frac{(-1)^{|\mathcal{H}|}}{1 - \epsilon_{(\mathcal{N} - (\mathcal{S} - \{i\})) \cup \mathcal{H}}} = \sum_{\substack{\mathcal{H} \subseteq \mathcal{S} - \{i\} \\ j \notin \mathcal{H}}} \frac{(-1)^{|\mathcal{H}|}}{1 - \epsilon_{\mathcal{N} - \mathcal{S} \cup i \cup \mathcal{H}}} + \sum_{\substack{\mathcal{H} \subseteq \mathcal{S} - \{i\} \\ j \in \mathcal{H}}} \frac{(-1)^{|\mathcal{H}|}}{1 - \epsilon_{\mathcal{N} - \mathcal{S} \cup i \cup j \cup \mathcal{H} - \{j\}}} \\ &= \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, j\}} \frac{(-1)^{|\mathcal{H}|}}{1 - \epsilon_{\mathcal{N} - \mathcal{S} \cup i \cup \mathcal{H}}} + \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, j\}} \frac{(-1)^{|\mathcal{H}|+1}}{1 - \epsilon_{\mathcal{N} - \mathcal{S} \cup i \cup j \cup \mathcal{H}}}. \end{aligned} \quad (80)$$

For an arbitrary set \mathcal{S} , define $m_S = \min \{k \in \mathcal{S}\}$, so that it suffices to show $f_S^{m_S} R_{m_S} \geq f_S^i R_i$ for all \mathcal{S} and $i \in \mathcal{S}$. Since it holds, by (79), $\frac{R_{m_S}}{R_i} \geq \frac{\epsilon_i}{\epsilon_{m_S}}$, we will prove the desired inequality $\frac{R_{m_S}}{R_i} \geq \frac{f_S^i}{f_S^{m_S}}$ by proving the stronger inequality $\frac{\epsilon_i}{\epsilon_{m_S}} \stackrel{?}{\geq} \frac{f_S^i}{f_S^{m_S}}$, or equivalently

$$\epsilon_i f_S^{m_S} \stackrel{?}{\geq} \epsilon_{m_S} f_S^i, \quad \forall \mathcal{S}, i \in \mathcal{S}. \quad (81)$$

We now concentrate on (81) and manipulate it through (80) to produce the equivalent relation

$$\begin{aligned} \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, m_S\}} (-1)^{|\mathcal{H}|} \frac{\epsilon_{m_S} - \epsilon_i}{1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_i \epsilon_{\mathcal{H}}} &\stackrel{?}{\geq} \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, m_S\}} (-1)^{|\mathcal{H}|} \left(\frac{\epsilon_{m_S}}{1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_i \epsilon_{\mathcal{H}}} - \frac{\epsilon_i}{1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_{\mathcal{H}}} \right) \\ \Leftrightarrow \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, m_S\}} (-1)^{|\mathcal{H}|} \frac{\epsilon_{m_S} - \epsilon_i}{1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_i \epsilon_{\mathcal{H}}} &\stackrel{?}{\geq} \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, m_S\}} (-1)^{|\mathcal{H}|} \frac{(\epsilon_{m_S} - \epsilon_i)(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_i \epsilon_{\mathcal{H}} - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_{\mathcal{H}})}{(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_i \epsilon_{\mathcal{H}})(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_{\mathcal{H}})}. \end{aligned} \quad (82)$$

Using the fact that $\epsilon_{m_S} \geq \epsilon_i$ and the following equality

$$\frac{1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_i \epsilon_{\mathcal{H}} - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_{\mathcal{H}}}{(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_i \epsilon_{\mathcal{H}})(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_{\mathcal{H}})} = 1 - \frac{\epsilon_{\mathcal{N}-\mathcal{S}}^2 \epsilon_{\mathcal{H}}^2 \epsilon_i \epsilon_{m_S}}{(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_i \epsilon_{\mathcal{H}})(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_{\mathcal{H}})}, \quad (83)$$

we can write an equivalent expression to (82) as

$$\sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, m_S\}} \frac{(-1)^{|\mathcal{H}|}}{1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_i \epsilon_{\mathcal{H}}} + \sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, m_S\}} (-1)^{|\mathcal{H}|} \frac{\epsilon_{\mathcal{N}-\mathcal{S}}^2 \epsilon_{\mathcal{H}}^2 \epsilon_i \epsilon_{m_S}}{(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_i \epsilon_{\mathcal{H}})(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_{\mathcal{H}})} \stackrel{?}{\geq} 0, \quad (84)$$

where we also used the identity $\sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, m_S\}} (-1)^{|\mathcal{H}|} = 0$.

We now observe that the first term of (84) is equal to the non-negative quantity $f_{\mathcal{S} - \{m_S\}}^i$ so that, in order to prove (84), it suffices to prove the second term in (84) to be non-negative, namely

$$\sum_{\mathcal{H} \subseteq \mathcal{S} - \{i, m_S\}} (-1)^{|\mathcal{H}|} \frac{\epsilon_{\mathcal{N}-\mathcal{S}}^2 \epsilon_{\mathcal{H}}^2 \epsilon_i \epsilon_{m_S}}{(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_i \epsilon_{\mathcal{H}})(1 - \epsilon_{\mathcal{N}-\mathcal{S}} \epsilon_{m_S} \epsilon_{\mathcal{H}})} \stackrel{?}{\geq} 0. \quad (85)$$

Eq. (85) is now a special case of the following general result

Lemma 13: For any $0 \leq \alpha_1, \alpha_2 < 1$, it holds

$$\sum_{\mathcal{H} \subseteq \mathcal{S}} (-1)^{|\mathcal{H}|} \frac{\prod_{i \in \mathcal{H}} \epsilon_i^2}{(1 - \alpha_1 \prod_{i \in \mathcal{H}} \epsilon_i)(1 - \alpha_2 \prod_{i \in \mathcal{H}} \epsilon_i)} \geq 0. \quad (86)$$

Proof: Using the geometric series $\sum_{l=0}^{\infty} z^l = 1/(1-z)$, for all $0 \leq z < 1$, and setting $z = \alpha_1 \prod_{i \in \mathcal{H}} \epsilon_i$ and $z = \alpha_2 \prod_{i \in \mathcal{H}} \epsilon_i$, yields

$$\begin{aligned} \frac{\prod_{i \in \mathcal{H}} \epsilon_i^2}{(1 - \alpha_1 \prod_{i \in \mathcal{H}} \epsilon_i)(1 - \alpha_2 \prod_{i \in \mathcal{H}} \epsilon_i)} &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \left(\alpha_1 \prod_{i \in \mathcal{H}} \epsilon_i \right)^l \left(\alpha_2 \prod_{i \in \mathcal{H}} \epsilon_i \right)^k \left(\prod_{i \in \mathcal{H}} \epsilon_i \right)^2 \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \alpha_1^l \alpha_2^k \prod_{i \in \mathcal{H}} \epsilon_i^{l+k+2}. \end{aligned} \quad (87)$$

Multiplying (87) with $(-1)^{|\mathcal{H}|}$, summing over all $\mathcal{H} \subseteq \mathcal{S}$ and using the identity $\prod_{i \in \mathcal{S}} (1 - x_i) = \sum_{\mathcal{H} \subseteq \mathcal{S}} (-1)^{|\mathcal{H}|} \prod_{i \in \mathcal{H}} x_i$ (which is easily proved by induction on $|\mathcal{S}|$) now produces

$$\text{LHS of (86)} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \alpha_1^l \alpha_2^k \sum_{\mathcal{H} \subseteq \mathcal{S}} (-1)^{|\mathcal{H}|} \prod_{i \in \mathcal{H}} \epsilon_i^{l+k+2} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \alpha_1^l \alpha_2^k \prod_{i \in \mathcal{S}} (1 - \epsilon_i^{l+k+2}) \geq 0, \quad (88)$$

which is the desired result. ■

APPENDIX F

CORRECTNESS OF CODE2_{pub}

The following result is a close analogue to Lemma 5.

Lemma 14: Consider a slot t in subphase 2.1 of CODE2_{pub}, when queues $Q_{\{i^*,j\}} \in \mathcal{Q}_{\dot{S}u}$ and $Q_{\{1,2,3\}}$ are combined, and a packet $s = \sum_{p \in Q_{\{i^*,j\}} \cup Q_{\{1,2,3\}}} a_s(p)p$ is transmitted. Assume that at the beginning of the slot (i.e. before any packet transmission), there exist sets $\mathcal{B}_{\mathcal{I}}^{(l)}(t) \subseteq \{\mathbf{b}_p^{(l)} : p \in Q_{\mathcal{I}}\}$, for all $\mathcal{I} \subseteq \mathcal{N}$ and $l \in \mathcal{I}$, and $\mathcal{B}_{D_l}(t) = \{\mathbf{b}_p^{(l)} : p \in Q_{D_l}\}$ such that $\mathcal{B}_{D_l}(t) \cup \bigcup_{\substack{\mathcal{I}: \mathcal{I} \subseteq \mathcal{N} \\ K_{\mathcal{I}}^l(t) > 0}} \mathcal{B}_{\mathcal{I}}^{(l)}(t)$ is a basis of $\mathbb{F}_q^{|\mathcal{K}_l|}$ for all $l \in \mathcal{N}$. Define $\mathcal{R}_{\{i^*,j\}}(t) \triangleq \left\{ l \in \{i^*,j\} : K_{\{i^*,j\}}^l(t) > 0 \vee K_{\{1,2,3\}}^l(t) > 0 \right\}$ and, for each $l \in \mathcal{R}_{\{i^*,j\}}(t)$, pick a vector $\hat{\mathbf{b}}_l$ as follows

$$\hat{\mathbf{b}}_l = \begin{cases} \text{arbitrary } \mathbf{b}_p^{(l)} \in \mathcal{B}_{\{1,2,3\}}^{(l)}(t) & \text{if } \dot{S}u(l) = 0, \\ \text{arbitrary } \mathbf{b}_p^{(l)} \in \mathcal{B}_{\{i^*,j\}}^{(l)}(t) & \text{otherwise.} \end{cases}$$

Then there exist coefficients $a_s(p)$ such that the set $\{\mathbf{b}_s^{(l)}\} \cup \mathcal{B}_{D_l}(t) \cup \bigcup_{\substack{\mathcal{I}: \mathcal{I} \subseteq \mathcal{N} \\ K_{\mathcal{I}}^l(t) > 0}} \mathcal{B}_{\mathcal{I}}^{(l)}(t)$ is a basis of $\mathbb{F}_q^{|\mathcal{K}_l|}$ for all $l \in \mathcal{R}_{\{i^*,j\}}(t)$.

Proof: The proof is essentially a repetition of the proof of Lemma 5, the main ingredients being the application of Lemma 9 to show that

$$\Pr \left(\left\{ \mathbf{b}_s^{(l)} \right\} \cup \mathcal{B}_{D_l}(t) \cup \bigcup_{\substack{\mathcal{I}: \mathcal{I} \subseteq \mathcal{N} \\ K_{\mathcal{I}}^l(t) > 0}} \mathcal{B}_{\mathcal{I}}^{(l)}(t) - \{\hat{\mathbf{b}}_l\} \text{ is basis of } \mathbb{F}_q^{|\mathcal{K}_l|} \right) \geq 1 - \frac{1}{q},$$

for all $l \in \mathcal{R}_{\{i^*,j\}}(t)$, and a standard probabilistic argument where $a_s(p)$ are selected iid uniformly in \mathbb{F}_q . ■

Lemma 14 can now be used to show that Lemma 6 is also true for CODE2_{pub}. This is again proved by induction on each slot t . In fact, since CODE2_{pub} is *identical* to CODE1_{pub} up to t_2^* (the time where each level 2 queue has at most one surviving index), it follows that the inductive hypothesis is true for all slots $t \leq t_2^*$, so we only need to apply induction for $t > t_2^*$. Due to the queue mixing in subphase 2.1, the proof of Lemma 6 must be modified as follows.

Proof of Lemma 6 for CODE2_{pub}: Assume that the inductive hypothesis holds at the beginning of slot $t > t_2^*$ and we are currently combining $Q_{\{i^*,j\}} \in \mathcal{Q}_{\dot{S}u}$ with $Q_{\{1,2,3\}}$. We pick the coefficients for the packet s to be transmitted at slot t according to Lemma 14 and distinguish the following mutually exclusive cases for each $l \in \mathcal{R}_{\{i^*,j\}}(t)$ (for $l \notin \mathcal{R}_{\{i^*,j\}}(t)$, the hypothesis holds for $t+1$ without changing any $\mathcal{B}_{\mathcal{I}}^{(l)}$, i.e. we simply select $\mathcal{B}_{\mathcal{I}}^{(l)}(t+1) = \mathcal{B}_{\mathcal{I}}^{(l)}(t)$)

- if l receives s and it holds $\dot{S}u(l) = 0$, ACTFB2 requires that $K_{\{1,2,3\}}^l$ is decreased by 1 and K_{D_l} is increased by 1. We set $\mathcal{B}_{\{1,2,3\}}^{(l)}(t+1) = \mathcal{B}_{\{1,2,3\}}^{(l)}(t) - \{\hat{\mathbf{b}}_l\}$ and $\mathcal{B}_{D_l}(t+1) = \mathcal{B}_{D_l}(t) \cup \{\mathbf{b}_s^{(l)}\}$ while all other sets remain unchanged. Lemma 14 implies that the new sets form a basis of $\mathbb{F}_q^{|\mathcal{K}_l|}$ at slot $t+1$.
- if l receives s and it holds $\dot{S}u(l) > 0$, then, according to ACTFB2, $K_{\{i^*,j\}}^l$ is decreased by one and K_{D_l} increased by 1. The hypothesis still holds for user l and slot $t+1$ by setting $\mathcal{B}_{D_l}(t+1) = \mathcal{B}_{D_l}(t) \cup \{\mathbf{b}_s^{(l)}\}$ and $\mathcal{B}_{\{i^*,j\}}^{(l)}(t+1) = \mathcal{B}_{\{i^*,j\}}^{(l)}(t) - \{\hat{\mathbf{b}}_l\}$, while all other sets remain unchanged.
- if l erases s and $k \in \{1, 2, 3\} - \{i^*, j\}$ receives it, CODE2_{pub} requires $K_{\{i^*,j\}}^l$ to be decreased by 1 and $K_{\{1,2,3\}}^l$ increased by one. The inductive hypothesis at $t+1$ is still true by setting $\mathcal{B}_{\{i^*,j\}}^{(l)}(t+1) = \mathcal{B}_{\{i^*,j\}}^{(l)}(t) - \{\mathbf{b}_s^{(l)}\}$ and $\mathcal{B}_{\{1,2,3\}}^{(l)}(t+1) = \mathcal{B}_{\{1,2,3\}}^{(l)}(t) \cup \{\mathbf{b}_s^{(l)}\}$.
- in all other cases, no $K_{\mathcal{I}}^l$, K_{D_l} indices change, so that sets $\mathcal{B}_{\mathcal{I}}^{(l)}$, \mathcal{B}_{D_l} remain the same as in slot t , and the hypothesis is trivially true at slot $t+1$.

Since the above list contains all possible cases, the inductive hypothesis always holds for all $l \in \mathcal{N}$ in slot $t+1$ and the proof is complete. \blacksquare

APPENDIX G

PROOF OF THEOREM 5

Consider a vector \mathbf{R} and assume without loss of generality that $\mathbf{R} > \mathbf{0}$. As in the analysis of CODE1_{pub}, we consider a modified version with a fixed blocklength n where the transmitter creates sets of packets \mathcal{K}_i with $|\mathcal{K}_i| = K_i(\mathbf{R}) = \lceil nR_i \rceil$, for $i \in \{1, 2, 3\}$, and transmits n symbols. An error is declared if CODE2_{pub} has not terminated by the n -th transmission. The proof is similar to that of Theorem 1, in the sense that the total number of slots \dot{T}^* required by CODE2_{pub} is computed as a random variable and it is seen that \dot{T}^*/n tends to a deterministic quantity $\bar{T}^*(\mathbf{R})$ w.p. 1 as $n \rightarrow \infty$, so that the achievable region of CODE2_{pub} is $\{\mathbf{R} : \bar{T}^*(\mathbf{R}) \leq 1\}$. Having found an exact expression for $\bar{T}^*(\mathbf{R})$, simple algebra reveals the latter region to be identical to the outer bound of Lemma 4.

We denote $\mathcal{N} = \{1, 2, 3\}$ while $\dot{T}_{\mathcal{S}}^*$ is the (random) number of time slots it takes CODE2_{pub} to process queue $Q_{\mathcal{S}}$, so that $\dot{T}^* = \sum_{\emptyset \neq \mathcal{S} \subseteq \mathcal{N}} \dot{T}_{\mathcal{S}}^*$. Since CODE2_{pub} is identical to CODE1_{pub} until the end of phase 2 (i.e. when each level 2 queue has at most one non-zero K index), we conclude that all level 1 queues are processed identically to CODE1_{pub}, so that Corollary 2 implies, through the appropriate substitutions

$$\lim_{n \rightarrow \infty} \sum_{\mathcal{S}: |\mathcal{S}|=1} \frac{\dot{T}_{\mathcal{S}}^*}{n} = \sum_{i \in \mathcal{N}} \hat{f}_{\{i\}}^i R_i = \frac{R_1 + R_2 + R_3}{1 - \epsilon_{\mathcal{N}}} \quad a.e. \quad (89)$$

We now make the following crucial observation regarding the decision taken by CODE2_{pub} at the end of phase 2 (denoted as t_2^*). Depending on the exact values of $\dot{S}u(i)$, the following cases exist:

- if $\dot{S}u(i) = 0$ for all $i \in \mathcal{N}$, or $\dot{S}u(i) = 1$ for all $i \in \mathcal{N}$, CODE2_{pub} continues mimicking CODE1_{pub} until the end of the algorithm. In this case, the asymptotic behavior of $\dot{T}_{\mathcal{S}}^*$ is obviously still governed by Corollary 2.
- otherwise, CODE2_{pub} deviates from CODE1_{pub} by further processing each level 2 queue $Q_{\mathcal{S}}$ in subphase 2.1 mentioned in Section VI. An inspection of the ACTFB2 procedure indicates that, during the combining of a level 2 queue $Q_{\mathcal{S}}$ with $Q_{\mathcal{N}}$, the actions regarding indices $K_{\mathcal{S}}^i$ are *identical* to ACTFB1 (in fact, the only difference between ACTFB1 and ACTFB2 lies in the handling of indices $K_{\mathcal{N}}^i$). Since each level 2 queue is still processed until all its K indices become zero, we conclude that, if we denote with $T_{\mathcal{S}}^*$ the *total* number of slots required for the processing of $Q_{\mathcal{S}}$ during phase 2 and subphase 2.1, Corollary 2 still holds. However, the value of $\dot{K}_{\mathcal{N}}^i$ at the beginning of phase 3 will be different than the corresponding value under CODE1_{pub} due to the interjection of subphase 2.1.

Denote with \tilde{t}_3 the beginning of phase 3, equivalently the end of phase 2 or subphase 2.1 (if the latter occurred). Since CODE2_{pub} again mimics CODE1_{pub} during phase 3, Corollary 2 implies, under the obvious substitutions, that

$$\lim_{n \rightarrow \infty} \frac{\dot{T}_{\mathcal{N}}^*}{n} = \max_{i \in \mathcal{N}} \left[\lim_{n \rightarrow \infty} \left(\frac{\dot{K}_{\mathcal{N}}^i(\tilde{t}_3)}{n} \right) \frac{1}{1 - \epsilon_i} \right] \quad a.e., \quad (90)$$

provided that the rightmost limit exists w.p. 1 (this will be shown later). It then follows that

$$\lim_{n \rightarrow \infty} \frac{\dot{T}^*}{n} = \sum_{\mathcal{S}: |\mathcal{S}| \leq 2} \max_{l \in \mathcal{S}} (\hat{f}_{\mathcal{S}}^l R_l) + \lim_{n \rightarrow \infty} \frac{\dot{T}_{\mathcal{N}}^*}{n} \quad a.e., \quad (91)$$

so that we hereafter concentrate on the computation of the last limit, which clearly depends on the specific decision at t_2^* .

Denote with $\dot{T}_{i,\mathcal{S}}^*$ the number of slots it takes CODE1_{pub} (or CODE2_{pub} , if we consider both phase 2 and subphase 2.1) to process a level 2 queue $Q_{\mathcal{S}}$ until $K_{\mathcal{S}}^i$ becomes 0. It clearly holds $\dot{T}_{\mathcal{S}}^* = \max_{i \in \mathcal{S}} \dot{T}_{i,\mathcal{S}}^*$; if we also define $\dot{T}_{\mathcal{S}}^\dagger = \min_{i \in \mathcal{S}} \dot{T}_{i,\mathcal{S}}^*$, we can combine Lemma 11 and Corollary 2 to deduce

$$\bar{T}_{\mathcal{S}}^\dagger \triangleq \lim_{n \rightarrow \infty} \frac{\dot{T}_{\mathcal{S}}^\dagger}{n} = \min_{i \in \mathcal{S}} (\hat{f}_{\mathcal{S}}^i R_i) \quad a.e., \quad (92)$$

in addition to

$$\begin{aligned} \bar{T}_{i,\mathcal{S}}^* &\triangleq \lim_{n \rightarrow \infty} \frac{\dot{T}_{i,\mathcal{S}}^*}{n} = \hat{f}_{\mathcal{S}}^i R_i \\ \bar{T}_{\mathcal{S}}^* &\triangleq \lim_{n \rightarrow \infty} \frac{\dot{T}_{\mathcal{S}}^*}{n} = \max_{i \in \mathcal{S}} (\hat{f}_{\mathcal{S}}^i R_i) \quad a.e., \end{aligned} \quad (93)$$

which we already used in (91).

We next find an expression for $\dot{K}_S^i(t_2^*)$, for all S with $|S| = 2$, since this will affect the branching decision made by CODE2_{pub} at t_2^* . The following relation is true for all S with $|S| = 2$ and describes the total decrease of each K index of a level 2 queue in the interval $[t_1^* \ t_2^*]$.

$$\dot{K}_S^i(t_2^*) = \dot{K}_S^i(t_1^*) - \sum_{l=1}^{\dot{T}_S^+} \mathbb{I}[K_S^i \text{ reduced by 1 during } l\text{-th slot of processing } Q_S \text{ in phase 2}]. \quad (94)$$

Dividing by n and using (92) we conclude that

$$\begin{aligned} k_{2,S}^i &\triangleq \lim_{n \rightarrow \infty} \frac{\dot{K}_S^i(t_2^*)}{n} = \left(\lim_{n \rightarrow \infty} \frac{\dot{K}_S^i(t_1^*)}{n} \right) - \left(\lim_{n \rightarrow \infty} \frac{\dot{T}_S^+}{n} \right) \Pr(K_S^i \text{ reduced by 1 during proc. } Q_S) \\ &= \hat{f}_S^i R_i (1 - \epsilon_{\mathcal{N}-(S-\{i\})}) - \min_{l \in S} (\hat{f}_S^l R_l) (1 - \epsilon_{\mathcal{N}-(S-\{i\})}) \quad a.e., \forall S : |S| = 2, \end{aligned} \quad (95)$$

where we used Lemma 11 (which is still applicable at t_1^*) for the asymptotic behavior of $\dot{K}_S^i(t_1^*)/n$. The subscript 2 emphasizes that the quantity refers to a limit of a random variable at t_2^* .

For $S = \{i, j\}$, (95) can be written as

$$\lim_{n \rightarrow \infty} \frac{\dot{K}_{\{i,j\}}^i(t_2^*)}{n} = k_{2,\{i,j\}}^i = \left[\hat{f}_{\{i,j\}}^i R_i - \hat{f}_{\{i,j\}}^j R_j \right]^+ (1 - \epsilon_{\mathcal{N}-\{j\}}) \quad a.e., \quad (96)$$

which motivates us to define

$$r_{\{i,j\}}^i(\mathbf{R}) = \left[\hat{f}_{\{i,j\}}^i R_i - \hat{f}_{\{i,j\}}^j R_j \right]^+, \quad (97)$$

where $[x]^+ \triangleq \max(x, 0)$ and we explicitly state the \mathbf{R} dependence of $r_{\{i,j\}}^i$. The binary relation $i \succ j$ is introduced to denote the inequality $\hat{f}_{\{i,j\}}^i R_i > \hat{f}_{\{i,j\}}^j R_j$ (equivalently, $r_{\{i,j\}}^i > 0$) which, using the definition of \hat{f}_S^i , can be expanded to

$$\hat{f}_{\{i,j\}}^i R_i > \hat{f}_{\{i,j\}}^j R_j \Leftrightarrow R_i \left(\frac{1}{1 - \epsilon_{\mathcal{N}-\{j\}}} - \frac{1}{1 - \epsilon_{\mathcal{N}}} \right) > R_j \left(\frac{1}{1 - \epsilon_{\mathcal{N}-\{i\}}} - \frac{1}{1 - \epsilon_{\mathcal{N}}} \right). \quad (98)$$

We also write $i \succeq j$ iff $\hat{f}_{\{i,j\}}^i R_i \geq \hat{f}_{\{i,j\}}^j R_j$ and $i \asymp j$ if $\hat{f}_{\{i,j\}}^i R_i = \hat{f}_{\{i,j\}}^j R_j$ (note that all relations \succ, \succeq, \asymp implicitly depend on \mathbf{R}), whence the following result follows.

Lemma 15: Consider any $\mathbf{R} > \mathbf{0}$ and distinct $i, j, k \in \mathcal{N}$. If $i \succ j$ and $j \succeq k$, then $i \succ k$. Similarly, if $i \succeq j$ and $j \succ k$, then $i \succ k$.

Proof: We prove by contradiction only the first part since the second one follows similarly. We assume that $k \succeq i$, so that it holds

$$\begin{aligned} \hat{f}_{\{i,j\}}^i R_i > \hat{f}_{\{i,j\}}^j R_j &\Leftrightarrow R_i \left(\frac{1}{1 - \epsilon_{\mathcal{N}-\{j\}}} - \frac{1}{1 - \epsilon_{\mathcal{N}}} \right) > R_j \left(\frac{1}{1 - \epsilon_{\mathcal{N}-\{i\}}} - \frac{1}{1 - \epsilon_{\mathcal{N}}} \right), \\ \hat{f}_{\{j,k\}}^j R_j &\geq \hat{f}_{\{j,k\}}^k R_k \Leftrightarrow R_j \left(\frac{1}{1 - \epsilon_{\mathcal{N}-\{k\}}} - \frac{1}{1 - \epsilon_{\mathcal{N}}} \right) \geq R_k \left(\frac{1}{1 - \epsilon_{\mathcal{N}-\{j\}}} - \frac{1}{1 - \epsilon_{\mathcal{N}}} \right), \\ \hat{f}_{\{i,k\}}^k R_k &\geq \hat{f}_{\{i,k\}}^i R_i \Leftrightarrow R_k \left(\frac{1}{1 - \epsilon_{\mathcal{N}-\{i\}}} - \frac{1}{1 - \epsilon_{\mathcal{N}}} \right) \geq R_i \left(\frac{1}{1 - \epsilon_{\mathcal{N}-\{k\}}} - \frac{1}{1 - \epsilon_{\mathcal{N}}} \right). \end{aligned} \quad (99)$$

The terms in parentheses above are non-negative by construction. In fact, the term $\frac{1}{1-\epsilon_{\mathcal{N}-\{j\}}} - \frac{1}{1-\epsilon_{\mathcal{N}}}$ is positive, since otherwise we would conclude that 0 is strictly larger than a non-negative number. We can then use a similar reasoning and the fact that $\mathbf{R} > \mathbf{0}$ to show that all terms in parentheses are positive. Hence, we can multiply the 3 equations by sides and arrive at a contradiction that a number is strictly larger than itself. \blacksquare

Using the notation of (43), we can find the value of $K_{\mathcal{N}}^i$ at t_2^* as

$$\dot{K}_{\mathcal{N}}^i(t_2^*) = \sum_{l=1}^{\lceil nR_i \rceil} \mathbb{I}[\dot{D}_{\{i\},l}^i = \mathcal{N}] + \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \sum_{l=1}^{\dot{T}_{\mathcal{S}}^{\dagger}} \mathbb{I}[\dot{D}_{\mathcal{S},l}^i = \mathcal{N}], \quad (100)$$

where the first, second term is the number of tokens moved during phase 1, 2, respectively. Using a procedure similar to Lemma 11, we can find

$$\begin{aligned} k_{2,\mathcal{N}}^i &\triangleq \lim_{n \rightarrow \infty} \frac{\dot{K}_{\mathcal{N}}^i(t_2^*)}{n} = R_i \Pr(\dot{D}_{\{i\}}^i = \mathcal{N}) + \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \left(\lim_{n \rightarrow \infty} \frac{\dot{T}_{\mathcal{S}}^{\dagger}}{n} \right) \Pr(\dot{D}_{\mathcal{S}}^i = \mathcal{N}) \\ &= \hat{f}_{\{i\}}^i R_i p_{\{i\},\mathcal{N}-\{i\}} + \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \min_{l \in \mathcal{S}} (\hat{f}_{\mathcal{S}}^l R_l) p_{i,\mathcal{N}-\mathcal{S}}. \end{aligned} \quad (101)$$

Any variation of $K_{\mathcal{N}}^i$ between t_2^* (end of phase 2) and \tilde{t}_3 (beginning of phase 3) under $\text{CODE}_{2_{pub}}$ can only be due to subphase 2.1 or the continuation of processing level 2 queues if $\dot{S}_u(l) = 1$ for all $l \in \mathcal{N}$.

Hence we conclude:

$$\dot{K}_{\mathcal{N}}^i(\tilde{t}_3) = \begin{cases} \dot{K}_{\mathcal{N}}^i(t_2^*) & \text{if } \dot{S}_u(l) = 0 \ \forall l \in \mathcal{N}, \\ \dot{K}_{\mathcal{N}}^i(t_2^*) + \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \sum_{l=1}^{\dot{T}_{i,\mathcal{S}}^* - \dot{T}_{\mathcal{S}}^{\dagger}} \mathbb{I}[\dot{D}_{\mathcal{S},l}^i = \mathcal{N}] & \text{if } \dot{S}_u(l) = 1 \ \forall i \in \mathcal{N}, \\ \left[\dot{K}_{\mathcal{N}}^i(t_2^*) + \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \sum_{l=1}^{\dot{T}_{i,\mathcal{S}}^* - \dot{T}_{\mathcal{S}}^{\dagger}} \mathbb{I}[\dot{D}_{\mathcal{S},l}^+] - \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \sum_{l=1}^{\dot{T}_{\mathcal{S}}^* - \dot{T}_{i,\mathcal{S}}^*} \mathbb{I}[\dot{D}_{\mathcal{S},l}^-] \right]^+ & \text{otherwise,} \end{cases} \quad (102)$$

where $\mathbb{I}[\dot{D}_{\mathcal{S},l}^+] \triangleq \mathbb{I}[K_{\mathcal{N}}^i \text{ increased during } l\text{-th slot of processing } Q_{\mathcal{S}} \text{ in subphase 2.1}]$ with a similar definition for $\mathbb{I}[\dot{D}_{\mathcal{S},l}^-]$ (replacing increased with decreased).

At this point, it is convenient to consider the following two complementary cases and individually examine each of them.

- it holds $r_{\mathcal{S}}^l = 0$ for all \mathcal{S} with $|\mathcal{S}| = 2$ and $l \in \mathcal{S}$. Equivalently, it holds $i \asymp j \asymp k$.
- it holds $r_{\mathcal{S}}^l > 0$ for at least one $l \in \mathcal{S}$ with $|\mathcal{S}| = 2$.

1) *The case $i \asymp j \asymp k$:* Equations (92), (93) imply that

$$\lim_{n \rightarrow \infty} \frac{\dot{T}_{\mathcal{S}}^* - \dot{T}_{i,\mathcal{S}}^*}{n} = \lim_{n \rightarrow \infty} \frac{\dot{T}_{i,\mathcal{S}}^* - \dot{T}_{\mathcal{S}}^{\dagger}}{n} = 0 \quad a.e., \quad (103)$$

so that, examining all 3 cases in (102), we conclude that

$$\lim_{n \rightarrow \infty} \frac{\dot{K}_{\mathcal{N}}^i(\tilde{t}_3)}{n} = \lim_{n \rightarrow \infty} \frac{\dot{K}_{\mathcal{N}}^i(t_2^*)}{n} \quad a.e., \quad (104)$$

which implies, through (90), (91), that CODE1_{pub} and CODE2_{pub} have the same asymptotic performance (meaning that $\bar{T}^*(\mathbf{R}) = \lim_{n \rightarrow \infty} \dot{T}^*/n$ is the same function under both algorithms) for all \mathbf{R} such that $i \asymp j \asymp k$. Hence, defining the set $\tilde{\mathcal{R}} \triangleq \{\mathbf{R} \geq \mathbf{0} : i \asymp j \asymp k\}$, we conclude $\mathcal{R}_{\text{CODE1}_{pub}} \cap \tilde{\mathcal{R}} = \mathcal{R}_{\text{CODE2}_{pub}} \cap \tilde{\mathcal{R}}$. Furthermore, it holds $\tilde{\mathcal{R}} \subseteq \mathcal{R}_{ord}$, where \mathcal{R}_{ord} was defined in (12), so that

$$\mathcal{C}^{out} \cap \tilde{\mathcal{R}} = \mathcal{C}^{out} \cap \mathcal{R}_{ord} \cap \tilde{\mathcal{R}} = \mathcal{R}_{\text{CODE1}_{pub}} \cap \mathcal{R}_{ord} \cap \tilde{\mathcal{R}} \subseteq \mathcal{R}_{\text{CODE2}_{pub}} \cap \tilde{\mathcal{R}}, \quad (105)$$

where the last set equality is due to Theorem 2. Hence, CODE2_{pub} achieves all rates in $\mathcal{C}^{out} \cap \tilde{\mathcal{R}}$.

2) *The case $r_S^l > 0$ for at least one $l \in \mathcal{S}$ with $|\mathcal{S}| = 2$:* Let $\mathcal{S} = \{i, j\}$ and assume $r_S^i > 0$, so that $i \succ j$. Then, two mutually exclusive cases exist according to Lemma 15 (in the following, i, j, k are distinct):

- it holds $k \succeq i$, so that $k \succ j$.
- it holds $i \succ k$.

In the first case, it follows from (96) that it holds w.p. 1

$$\lim_{n \rightarrow \infty} \frac{\dot{K}_{\{i,j\}}^i(t_2^*)}{n} > 0, \quad \lim_{n \rightarrow \infty} \frac{\dot{K}_{\{i,j\}}^j(t_2^*)}{n} = 0, \quad (106)$$

$$\lim_{n \rightarrow \infty} \frac{\dot{K}_{\{j,k\}}^k(t_2^*)}{n} > 0, \quad \lim_{n \rightarrow \infty} \frac{\dot{K}_{\{j,k\}}^j(t_2^*)}{n} = 0, \quad (107)$$

so that, but the definition of limit, there exists some n_1 such that for all $n > n_1$ it holds $\dot{S}u(i) \geq 1$, $\dot{S}u(k) \geq 1$, $\dot{S}u(j) = 0$. In the second case, (106) is still true and it also holds

$$\lim_{n \rightarrow \infty} \frac{\dot{K}_{\{i,k\}}^i(t_2^*)}{n} > 0, \quad \lim_{n \rightarrow \infty} \frac{\dot{K}_{\{i,k\}}^k(t_2^*)}{n} = 0, \quad (108)$$

which implies via a similar argument that there exists some n_2 such that $\dot{S}u(i) = 2$, $\dot{S}u(j) \leq 1$, $\dot{S}u(k) \leq 1$, for all $n > n_2$.

Hence, in both cases there exists a sufficiently large n_0 such that for all $n > n_0$, the first two branches in (102) are excluded. Hence, it holds

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\dot{K}_{\mathcal{N}}^i(\tilde{t}_3)}{n} = & \left[\lim_{n \rightarrow \infty} \frac{\dot{K}_{\mathcal{N}}^i(t_2^*)}{n} + \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \left(\hat{f}_S^i R_i - \min_{l \in \mathcal{S}} (\hat{f}_S^l R_l) \right) p_{\{i\}, \mathcal{N}-\mathcal{S}} \right. \\ & \left. - \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \left(\max_{l \in \mathcal{S}} (\hat{f}_S^l R_l) - \hat{f}_S^i \right) (1 - \epsilon_i) \right]^+ \quad a.e., \end{aligned} \quad (109)$$

which can also be written as

$$\lim_{n \rightarrow \infty} \frac{\dot{K}_{\mathcal{N}}^i(\tilde{t}_3)}{n} = \left[\lim_{n \rightarrow \infty} \frac{\dot{K}_{\mathcal{N}}^i(t_2^*)}{n} + \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \mathbb{I}[r_{\mathcal{S}}^i > 0] p_{\{i\}, \mathcal{N}-\mathcal{S}} - \sum_{\substack{\mathcal{S}: i \in \mathcal{S} \\ |\mathcal{S}|=2}} \mathbb{I}[r_{\mathcal{S}}^i = 0] (1 - \epsilon_i) \right]^+. \quad (110)$$

It is now a matter of case distinction, depending on the values of $r_{\mathcal{S}}^i$, and simple algebra to verify that $\text{CODE}_{2_{pub}}$ also achieves all rates in $\mathcal{C}^{out} \cap \tilde{\mathcal{R}}^c$, so that it achieves \mathcal{C}^{out} .

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